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In this paper we consider the location of the zeros of a complex polynomial $f(z)$ expressed as $f(z) = \sum_{k=0}^{n} a_k p_k(z)$ where $\{p_k(z)\}$ is a given sequence of polynomials of degree k whose zeros lie in a prescribed region E. The principal theorem states that the zeros of $f(z)$ are in the interior of a Jordan curve $S = \{z; \mid F(z) \mid = \text{Max}\,(1, R)\}$ where F maps the complement of E onto $\mid z \mid > 1$ and R is the positive root of the equation $\sum_{k=0}^{n-1} \lambda_k \mid a_k \mid t^k - \lambda n \mid a_n \mid t^n = 0$, with $\lambda_k > 0$ depending on E only. Several applications of this theorem are given. For example; if $\{p_k(z)\}$ is a sequence of orthogonal polynomials on $a \leqq z \leqq b$, then we give an ellipse containing all the zeros of $\sum_{k=0}^{n} a_k p_k(z)$.

Previous results. An extensive mathematical literature deals with the location of the zeros in the complex plane of a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{1}$$

with complex coefficients a_j. Cauchy derived practical bounds for the moduli of the zeros of (1) using the moduli of the coefficients a_j. In many investigations the polynomial (1) is not expressed as a linear combination of the sequence $\{z^k\}$, but as

$$f(z) = b_0 + b_1 p_1(z) + \cdots + b_n p_n(z) \tag{2}$$

where $\{p_k(z)\}$ is a given sequence of polynomials. Cauchy's well known result (Marden [2], Th. 27, 1) was generalized by Turán [4] in the case where the expansion in (2) is the Hermite-expansion $e^{z^2} \sum_{k=0}^{n} b_k (e^{-z^2})^{(k)}$. He obtained upper bounds for the moduli of the imaginary parts of the zeros, i.e., a "strip" where all the zeros of (2) are located. Specht [3], making use of the Christoffel-Darboux formula, extended these results to other sequences of orthogonal polynomials. In our Theorem 1, we replace the "strip" with a bounded region, which will yield an ellipse in the case where the $\{p_k(z)\}$ is a sequence of orthogonal polynomials on a finite interval.

2. Cauchy type estimate. In the sequal we shall use the following notations: Let E be a compact (infinite) set in the complex z-plane, whose complement G is simply connected, $w = F(z)$ the univalent function which is defined on G and maps G conformally on D: $\mid w \mid > 1$ such that the point at infinity in the two planes correspond to each other and also preserves direction there. The function $F(z)$ has the expansion in the point at infinity

$$w = F(z) = \frac{1}{\tau} z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots \tag{3}$$

where τ is the transfinite diameter of E. If the boundary B of G is a Jordan curve, then according to a well known theorem (see Carathéodory [1]) the function $F(z)$ is continuous in the closure of G and maps the boundary B one-to-one onto $|w| = 1$. We shall denote by C_R the inverse image of the circle $|F(z)| = R$ $(R > 1)$.

With these notations we are able to state the following:

THEOREM 1. *Let G be the complement of a finite domain E whose boundary B is a Jordan curve. If $\mathscr{P} = \{p_r(z); E\}$ is a sequence of polynomials of exact degree r whose zeros are in E, then the polynomial*

$$f(z) = a_0 p_0 + a_1 p_1(z) + \cdots + a_n p_n(z) \tag{4}$$

has its zeros in the closed interior of the Jordan curve $S = \{z; |F(z)| = \mathrm{Max}(1, R)\}$, where R is the (only) positive root of the equation

$$\lambda_0 |a_0| + \lambda_1 |a_1| t + \cdots + \lambda_{n-1} |a_{n-1}| t^{n-1} - \lambda_n |a_n| t^n = 0 \,. \tag{5}$$

The λ_r are positive and depend only on $\mathscr{P}$ and E.

Proof. The rational function $(p_r(z)/p_{r+1}(z))$ has the expansion in the neighborhood of the point at infinity:

$$\frac{p_r(z)}{p_{r+1}(z)} = b_1 z^{-1} + b_2 z^{-2} + \cdots \qquad (b_1 \neq 0) \,. \tag{6}$$

Define

$$g_r(z) = \frac{p_r(z)}{p_{r+1}(z)} F(z) \,. \tag{7}$$

Using (6) and (3) we obtain the following expansion for g_r at $z = \infty$:

$$g_r(z) = d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots \qquad (d_0 \neq 0) \,.$$

Hence $g_r(z)$ is analytic in the domain G and continuous in $G \cup B$. With the aid of the maximum modulus theorem we obtain:

$$|g_r(z)| \leqq \mathop{\mathrm{Max}}_{z \in B} \left| \frac{p_r(z)}{p_{r+1}(z)} \right| |F(z)| = \mathop{\mathrm{Max}}_{z \in B} \left| \frac{p_r(z)}{p_{r+1}(z)} \right| = m_r \,, \tag{8}$$

since $|F(z)| = 1$ or B. From (7) and (8) we obtain the estimate

$$\left|\frac{p_r(z)}{p_{r+1}(z)}\right| \leqq \frac{m_r}{|F(z)|} \quad \text{for} \quad z \in G\,. \tag{9}$$

For $r < n$

$$\left|\frac{p_r(z)}{p_n(z)}\right| = \left|\frac{p_r(z)}{p_{r+1}(z)}\right| \left|\frac{p_{r+1}(z)}{p_{r+2}(z)}\right| \cdots \left|\frac{p_{n-1}(z)}{p_n(z)}\right| \leqq \frac{m_r m_{r+1} \cdots m_{n-1}}{|F(z)|^{n-r}}\,. \tag{10}$$

Denote $\lambda_r = m_r m_{r+1} \cdots m_{n-1}$, then for $z \in G$

$$\left|\frac{p_r(z)}{p_n(z)}\right| \leqq \frac{\lambda_r}{|F(z)|^{n-r}}\,. \tag{11}$$

Now, let $\zeta \in G$ be a zero of the polynomial in (4), then

$$|a_n|\,|p_n(\zeta)| \leqq |a_0| + |a_1|\,|p_1(\zeta)| + \cdots + |a_{n-1}|\,|p_{n-1}(\zeta)| \tag{12}$$

from which, after dividing by $p_n(\zeta) \neq 0$ and using (11), we obtain

$$|a_n|\,|F(\zeta)|^n \leqq \lambda_0 |a_0| + \lambda_1 |a_1|\,|F(\zeta)| + \cdots + \lambda_{n-1} |a_{n-1}|\,|F(\zeta)|^{n-1}\,. \tag{13}$$

But this inequality implies that $|F(\zeta)| \neq R$, for R is the root of (5). From the definition of C_R it follows that ζ is in the closed interior of C_R. If $\zeta \in E$ then clearly ζ is in the interior of C_1, hence all the zeros of (4) lie in the closed interior of the Jordan curve

$$S = \{z\colon |F(z)| = \operatorname{Max}(1, R)\}\,,$$

which proves the theorem.

As an application of this theorem, consider a sequence of polynomials $p_r(z)$ with leading coefficient one and whose zeros (which we assume lie in the interval $[-1, +1]$) separate each other. More precisely: if $z_{1,r}, z_{2,r}, \cdots, z_{r,r}$ are the zeros of $p_r(z)$ and $z_{0,r} = -1$, $z_{r+1,r} = 1$, then each interval $(z_{k,r}, z_{k+1,r})$, $k = 0, 1, \cdots, r$, contains exactly one zero of $p_{r+1}(z)$. The mapping function which maps the exterior of $[-1, +1]$ onto the exterior of the unit circle is given by

$$w = F(z) = z + (z^2 - 1)^{1/2} \tag{14}$$

where we take that branch of $z + (z^2 - 1)^{1/2}$ which becomes infinite at $z = \infty$. The locus $C_R = \{z; |w| = R\}$ will be an ellipse with foci at $+1, -1$ and with semi axes $(1/2)(R + R^{-1})$, $(1/2)(R - R^{-1})$. Now, if $R = 2 + 3^{1/2}$ then C_R is the ellipse with major axis 4 and minor axis $2 \cdot 3^{1/2}$. The distance of any point u outside or on C_R from the zeros of $p_r(z)$ $(r = 1, 2, \cdots)$ is greater than 1. Let u be such a point; then

$$\left|\frac{p_r(u)}{p_{r+1}(u)}\right| = \left|\frac{(u - z_{1,r})(u - z_{2,r}) \cdots (u - z_{r,r})}{(u - z_{1,r+1}) \cdots (u - z_{r+1,r+1})}\right| \leqq 1 ; \tag{15}$$

for suppose that the minimum distance of u from $z_{k,r+1}$ is attained at $z_{k_0,r+1}$, i.e.,

$$|u - z_{k_0,r+1}| = \min_{1 \leqq k \leqq r+1} |u - z_{k,r+1}| > 1 . \tag{16}$$

If we replace $z_{k,r}$ by $z_{k+1,r+1}$ when $k \geqq k_0$ and $z_{k,r}$ by $z_{k,r+1}$ when $k < k_0$ in (15), then the numerator is increased because the zeros are separated. Using (16) we obtain

$$\left|\frac{p_r(u)}{p_{r+1}(u)}\right| \leqq \frac{1}{|u - z_{k_0,r+1}|} \leqq 1 . \tag{17}$$

If the interior of $C_{2+3^{1/2}}$ is our domain E in Theorem 1, then it follows from (17) that all the λ_r satisfy $0 < \lambda_r < 1$ $r = 0, 1, 2, \cdots$ and we obtain the following:

THEOREM 2. *Let* $\{p_r(z)\}$ *be a sequence of polynomials with leading coefficient* 1. *If all the zeros of* $p_r(z)$ $(r = 0, 1, 2, \cdots)$ *lie in* $[-1, +1]$ *and the zeros of* $p_r(z)$ *and* $p_{r+1}(z)$ *separate each other, then all the zeros of the polynomial*

$$f(z) = \sum_{r=0}^{n} a_r p_r(z) \tag{18}$$

are in the ellipse

$$\frac{x^2}{(R + R^{-1})^2} + \frac{y^2}{(R - R^{-1})^2} = 1/4 \qquad (z = x + iy) \tag{19}$$

where $R = \max(2 + 3^{1/2}, \rho)$ *and* ρ *is the* (*only*) *positive root of*

$$|a_0| + |a_1| t + |a_2| t^2 + \cdots + |a_{n-1}| t^{n-1} - |a_n| t^n = 0 .$$

In particular, if the sequence $\{p_r(z)\}$ in Theorem 2 is a sequence of orthogonal polynomials then the zeros of $p_r(z)$ and $p_{r+1}(z)$ separate each other and we have, for example, the following:

COROLLARY. *If* $f(z) = \sum_{r=0}^{n} a_r p_r(z)$ *is a polynomial expounded in Legendre polynomials* $p_r(z)$, *then all the zeros of* $f(z)$ *are in the ellipse as given in Theorem* 2.

We will use Theorem 1 to prove a result, Theorem 3, which is analogous to Pellet's theorem (Marden [2], Th. 28, 1). Keeping the notation of Theorem 1, define:

$$d_r^{(k)} = \operatorname*{Max}_{z \in B} \left| \frac{p_r(z)}{p_k(z)} \right| \qquad (r = 0, 1, 2, \cdots) \tag{20}$$

then, as in (9)

$$\left| \frac{p_r(z)}{p_k(z)} \right| \leqq \frac{d_r^{(k)}}{| F(z) |^{k-r}} \quad \text{for} \quad z \in G\,. \tag{21}$$

With the aid of inequality (21) we prove:

THEOREM 3. *Let E, G, B and $\{p_r(z)\}$ be as in Theorem* 1. *If for a polynomial*

$$f(z) = \sum_{r=0}^{n} a_r p_r(z)\,. \quad a_k \neq 0 \textit{ for some } k, \quad 0 < k \leqq n \tag{22}$$

the equation:

$$H(t) = \sum_{r=0}^{k-1} d_r^{(k)} | a_r | t^r - d_k^{(k)} | a_k | t^k + \sum_{r=k+1}^{n} d_r^{(k)} | a_r | t^r = 0 \tag{23}$$

has two positive roots ρ and R[1] $1 < \rho < R$, *and if the only positive root of*

$$\sum_{r=0}^{k=1} d_r^{(k)} | a_r | t^r - d_k^{(k)} | a_k | t^k = 0 \tag{24}$$

is greater or equal to 1, *then $f(z)$ has exactly k zeros in or on C_ρ and no zeros in the open ring* Ext. $C_\rho \cap$ Int. C_R.

Proof. The region Ext. $C_\rho \cap$ Int. C_R, by assumption ($\rho > 1$) is contained in G, hence we will show that if $\zeta \in G$ is a zero of (22) then $\zeta \notin$ Ext. $C_\rho \cap$ Int. C_R. Because $p_r(\zeta) \neq 0$ and $\sum_{r=0}^{n} a_r p_r(\zeta) = 0$ we have:

$$| a_k | \leqq \sum_{\substack{r=0 \\ r \neq k}}^{n} | a_r | \left| \frac{p_r(\zeta)}{p_k(\zeta)} \right|\,. \tag{25}$$

Using inequality (21) we obtain

$$| a_k | \leqq \sum_{\substack{r=0 \\ r \neq k}}^{n} | a_r | \frac{d_r^{(k)}}{| F(\zeta) |^{k-r}} \tag{26}$$

and hence

$$\begin{aligned} H(| F(\zeta) |) = & \sum_{r=0}^{k-1} d_r^{(k)} | a_r | \, | F(\zeta) |^r \\ & - d_k^{(k)} | a_k | \, | F(\zeta) |^k + \sum_{r=k+1}^{n} d_r^{(k)} | a_r | \, | F(\zeta)^r | \geqq 0\,. \end{aligned} \tag{27}$$

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[1] If not all the $a_r (r < k)$ are zero, then the equation (23), according to the Descartes rule of sign, has two positive roots or has no positive roots at all.

This last inequality tells us that $t = |F(\zeta)| < \rho$ or $t = |F(\zeta)| > R$, because the function $H(t)$ is negative only for $\rho < t < R$. We have shown that $\zeta \notin C_t$ when $\rho < t < R$ i.e., $\zeta \notin \text{Ext.}\, C_\rho \cap \text{Int.}\, C_R$.

To complete the proof of Theorem 3 we have to show that the closed interior of C_ρ contains exactly k zeros. We will do it by using a continuity argument. Define

$$f(z; s) = \sum_{r=0}^{k} a_r p_r(z) + \sum_{r=k+1}^{n} s a_r p_r(z) \qquad (0 \leqq s \leqq 1) \tag{28}$$

and the function

$$H(t; s) = \sum_{r=0}^{k-1} d_r^{(k)} |a_r| t^r - d_k^{(k)} |a_k| t^k + \sum_{r=k+1}^{n} d_r^{(k)} s |a_r| t^r . \tag{29}$$

$H(t; s)$, if $s = 1$, has two positive roots $\rho(1) = \rho$, $R(1) = R$. If s tends to 0, the two roots $\rho(s)$ and $R(s)$ of (29) start to move; $\rho(s)$ decreases and $R(s)$ increases, and according to the first part of this theorem, $\text{Ext.}\, C_{\rho(s)} \cap \text{Int.}\, C_{R(s)}$ is always zero free, hence the number of zeros $N(s)$ in the closed interior $C_{\rho(s)}$ $(0 \leqq s \leqq 1)$ is a constant call it N. If $s \to 0$ then $R(s) \to \infty$ and $\rho(s) \to \rho^* \geqq 1$ by (24). But Theorem 1 applies to $f(z; 0)$ and $H(t; 0)$. Consequently the number of zeros in C_ρ^* is exactly k, i.e., $N = k$ which completes the proof of Theorem 3.

In the preceding theorems we obtained bounds for all the zeros of (2) as function of all the coefficients a_r. However, if we restrict ourselves to $p + 1$ fixed coefficients and $n - p$ arbitrary ones, are we able to find some bounds for p zeros? In the case $f(z) = \sum_{r=0}^{n} a_r z^r$ Van Vleck [5] proved that essentially there is only one case in which bounds for p zeros are derivable, i.e., if the fixed coefficients are the first p consecutive ones and any other one from the remaining set. In other words, he showed that if one of the coefficients $a_0, a_1, \cdots, a_{p-1}$ is arbitrary, then at least $n - p + 1$ zeros of $\sum_{r=0}^{n} a_r z^r$ may be made arbitrarily large in modulus. Perhaps it is interesting to note that this is the case in which the polynomial (2) is expressed with the aid of the sequence $p_n(z)$ in Theorem 1. Suppose a_k for some $k (0 \leqq k \leqq p-1)$ in $f(z) = \sum_0^n a_r p_r(z)$ is arbitrary. Let $\mathscr{L} > 1$ be such that the distance of $C_{\mathscr{L}}$ from the origin is greater than a fixed large number δ. Choose a_k so large in modulus that the equation in (24) has a root greater than 1 and also

$$d_k^{(k)} |a_k| \mathscr{L}^k < \sum_{\substack{r=0 \\ r \neq k}}^{n} d_r^{(k)} |a_r| \mathscr{L}^r . \tag{30}$$

But (30) implies that the equation $H(t) = 0$ in (23) has a root $R > \mathscr{L}$. According to Theorem 3 $n - k \geqq n - p + 1$ zeros $z_1, z_2, \cdots, z_{n-k}$ of

$f(z)$ are in Ext. C_R. Thus $|z_j| > \delta$ for $j = 1, 2, \cdots, n-k$, because Ext. $C_R \subset$ Ext. $C_{\mathscr{D}}$.

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