

Pacific Journal of Mathematics

**ON THE LATTICE OF CLOSED SUBSPACES OF HILBERT
SPACE**

NEAL ZIERLER

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The purpose of this note is to answer two questions which have arisen in connection with the lattice-theoretic characterization of the set of closed subspaces of a Hilbert space of countably infinite dimension which appears in "Axioms for nonrelativistic quantum mechanics," Pacific Journal of Mathematics, Vol. 11, No. 3, 1961, pages 1151-1169.

The material in this section is to replace [2, p. 1165, lines 10-32]. Up to that point it has been shown that the lattice under each finite element of P is isomorphic to the lattice of subspaces of a Hilbert space over a field D which is either real or complex. The orthocomplementation induced in a Hilbert space by such an isomorphism gives rise to an involution of D (vide infra). In this section we show that such an involution is continuous, thereby closing a gap brought to our attention by a comment of M. D. Maclaren.

Let $a \in P_f$ with $n = \dim a > 0$. Choose pairwise orthogonal points A_0, \dots, A_n in (a) and in each line $l_i = A_0 \vee A_i, i = 1, \dots, n$, choose, a point E_i different from A_0 and A_i . Clearly the points A_0, E_1, \dots, E_n are independent and the choice of $A_1 \vee \dots \vee A_n$ as improper hyperplane, A_0 as origin and E_1, \dots, E_n as unit points leads to the unique introduction of homogeneous coordinates in (a) in standard fashion. In particular, the proper points of l_1 are precisely those with homogeneous coordinates $(1, \lambda, 0, \dots, 0)$ which we abbreviate as $(1, \lambda) - \lambda$, of course, being any member of the field D that has been constructed. The topology for D is obtained as follows: The subset N of D is a neighborhood of 0 if $\{(1, \nu): \nu \in N\}$ is a neighborhood of A_0 in l_1 . Under this topology, D is either the real or complex field (cf. [2, Lemma 2.11 et seq., p. 1164]).

It is shown in [1] that there then exist an involution σ of D and numbers (= members of D) η_0, \dots, η_n such that

- (1) $\eta_i^\sigma = \eta_i$,
- (2) $\sum x_i \eta_i x_i^\sigma = 0$ if and only if all $x_i = 0$,
- (3) If $(x_0, \dots, x_n) \in (a)_0$, then $a(x_0, \dots, x_n)'$ (the complement of (x_0, \dots, x_n) in (a)) = $\vee \{(y_0, \dots, y_n) \in (a)_0: \sum y_i \eta_i x_i^\sigma = 0\}$

Note that by (2), no η_i is 0 and that $1, \eta_1/\eta_0, \dots, \eta_n/\eta_0$ defines the same orthomorphmentation as η_0, \dots, η_n ; i.e., we may assume that $\eta_0 = 1$.

Again confining our attention to l_1 , observe that if $\lambda \neq 0$ and $l_1(1, \lambda)'$ (the point of l_1 orthogonal to the point $(1, \lambda)$) has coordinates

$(1, \mu)$, then $\mu = -1/\eta_1 \lambda^\sigma$. Hence if $\lambda_m \rightarrow 1$ and is never 0, $(1, \lambda_m) \rightarrow (1, 1)$ by definition (of the topology for D) so $(1, \lambda_m)' \rightarrow (1, 1)' = (1, -1/\eta_1)$ by [2, Lemma 2.8]. But $(1, \lambda_m)' = (1, \mu_m)$ with $\mu_m = -1/\eta_1 \lambda_m^\sigma$. Then $(1, \mu_m) \rightarrow (1, -1/\eta_1)$ which implies $\mu_m \rightarrow -1/\eta_1$; i.e., $-1/\eta_1 \lambda_m^\sigma \rightarrow -1/\eta_1$ so $\lambda_m^\sigma \rightarrow 1$. Thus, σ is continuous at 1 and hence is continuous (if $\lambda_m \rightarrow 0$ then $\lambda_m + 1 \rightarrow 1$ so $(\lambda_m + 1)^\sigma = \lambda_m^\sigma + 1 \rightarrow 1$ so $\lambda_m^\sigma \rightarrow 0$). Of course, this result was automatic in the real case. It follows that σ is either the identity or, in the complex case, conjugation. It follows now from (2) that η_1, \dots, η_n are positive real numbers. If D is the complex numbers, σ is conjugation, for otherwise $(1, i\eta_1^{-1/2}, 0, \dots, 0)$ would be self-orthogonal.

Taking the Hilbert space of $n + 1$ tuples of D as H_a , the mapping $(x_0, \dots, x_n) \rightarrow \{\lambda(x_0, \dots, x_n): \lambda \in D\}$ clearly induces a continuous isomorphism φ_a of (a) on the lattice L_a of subspaces of H_a such that the orthocomplementation induced by φ_a in L_a is obtained from the inner product $(x, y) = \sum x_i \eta_i \bar{y}_i$ for H_a .

2. The following is a replacement for [2, p. 1165, lines 33 to 41]. Its purpose is to insure that all the isometries $\psi_{b,a}$ are linear rather than conjugate linear. I am indebted to V. S. Varadarajan for calling my attention to this omission.

Let $a \leq b$ be finite and suppose that, in accordance with what has preceded, we have selected a Hilbert space H_a over D of dimension $1 + \dim a$ and a continuous isomorphism φ_a of (a) on the lattice L_a of subspaces of H_a which is orthogonality-preserving in the sense that

$$(13) \quad \varphi_a(c) \perp \varphi_a(d) \text{ if and only if } c \perp d .$$

Suppose that H_b, φ_b , have been similarly chosen for b .

Now $\varphi_b \varphi_a^{-1}$ is a continuous, orthogonality-preserving isomorphism of L_a in L_b . Hence, as is well-known and not difficult to show, there exists a continuous automorphism σ of D and a σ -isometry $\psi_{b,a}$, unique up to multiplication by a number of modulus one, providing $\dim a > 0$ (see below), such that $\psi_{b,a}$ induces $\varphi_b \varphi_a^{-1}$ in the sense that $\varphi_b \varphi_a^{-1}[v] = [\psi_{b,a} v]$ for all $v \in H_a$, where $[v]$ denotes the linear subspace generated by v . A σ -isometry ψ of H is a mapping of H in itself with the following three properties:

$$(14) \quad \begin{aligned} \text{Additivity: } & \psi(u + v) = \psi(u) + \psi(v) \\ \sigma\text{-linearty: } & \psi(\lambda u) = \lambda^\sigma \psi(u) \\ \sigma\text{-isometry: } & (\psi(u), \psi(v)) = (u, v)^\sigma . \end{aligned}$$

A σ -isometry is said to be *linear* or *conjugate-linear* when σ is the identity or conjugation respectively.

If D is the real field, the automorphism σ is the identity, while

in the complex case, in view of its continuity, σ may be either the identity or conjugation. Observe that if $\dim a = 0$ and u, v are unit vectors in $H_a, \varphi_b(a)$ respectively, then $\lambda u \rightarrow \lambda v$ and $\lambda u \rightarrow \bar{\lambda}v$ both induce the mapping $\varphi_b\varphi_a^{-1}$ of L_a in L_b . In other words, $\psi_{b,a}$ may be chosen both linear and conjugate-linear when $\dim a = 0$, independent of the choice of H_a, φ_a and H_b, φ_b . In general, the linearity of $\psi_{b,a}$ may be achieved through the proper choice of H_b, φ_b as follows. Suppose that $\psi_{b,a}$ inducing $\varphi_b\varphi_a^{-1}$ is conjugate-linear. Let $\{v_i\}$ be a complete orthonormal set for H_b and define $\gamma: H_b \rightarrow H_b$ by: $\gamma(\sum \lambda_i v_i) = \sum \bar{\lambda}_i v_i$. Let φ denote the automorphism of L_b induced by γ and let $\bar{\varphi}_b = \varphi \circ \varphi_b$. Then $\bar{\varphi}_b$ is a continuous, orthogonality-preserving isomorphism of (b) on L_b which is induced by the linear isometry $\bar{\psi}_{b,a} = \gamma \circ \psi_{b,a}$.

Suppose now that $\dim a > 0$, that H_a, φ_a have been chosen arbitrarily and that for every finite $b > a$, H_b, φ_b has been chosen as above so that $\varphi_b\varphi_a^{-1}$ is "linear" in the sense that every isometry of H_a in H_b which induces it is linear. For each finite $c \not\geq a$ let $H_c = \varphi_{a \vee c}(c)$ and let $\varphi_c = \varphi_{a \vee c} | (c)$. Then $\varphi_{a \vee c}\varphi_c^{-1}$ is linear, for it is induced by the projection in $H_{a \vee c}$ of its subspace H_c .

Now that H_c, φ_c have been assigned to every finite c , it remains to show that $\varphi_{c_1}\varphi_{c_2}^{-1}$ is in fact linear whenever $c_2 < c_1$. The type of argument we shall use involves the introduction of $c_3 < c_2$ for which both $\varphi_{c_1}\varphi_{c_3}^{-1}$ and $\varphi_{c_2}\varphi_{c_3}^{-1}$ are known to be linear. The linearity of $\varphi_{c_1}\varphi_{c_2}^{-1}$ then follows from the equation $\varphi_{c_1}\varphi_{c_2}^{-1} = (\varphi_{c_1}\varphi_{c_3}^{-1})(\varphi_{c_2}\varphi_{c_3}^{-1})$.

Given finite $c_2 < c_1$, let $b_i = c_i \vee a, i = 1, 2$. Now $b_2 \leq b_1$ and $\varphi_{b_1}\varphi_{b_2}^{-1}$ is linear, for $\varphi_{b_i}\varphi_a^{-1}, i = 1, 2$ are linear by construction and $\varphi_{b_1}\varphi_a^{-1} = (\varphi_{b_1}\varphi_{b_2}^{-1})(\varphi_{b_2}\varphi_a^{-1})$. Since $\varphi_{b_1}\varphi_{b_2}^{-1}$ is linear and $\varphi_{b_2}\varphi_{c_2}^{-1}$ is linear by construction, $\varphi_{b_1}\varphi_{c_2}^{-1} = (\varphi_{b_1}\varphi_{b_2}^{-1})(\varphi_{b_2}\varphi_{c_2}^{-1})$ is linear. Finally, since $\varphi_{b_1}\varphi_{c_2}^{-1}$ is linear and $\varphi_{b_1}\varphi_{c_1}^{-1}$ is linear by construction, the linearity of $\varphi_{c_1}\varphi_{c_2}^{-1}$ follows from the equation $\varphi_{b_1}\varphi_{c_2}^{-1} = (\varphi_{b_1}\varphi_{c_1}^{-1})(\varphi_{c_1}\varphi_{c_2}^{-1})$.

Thus, each finite c has been provided with H_c, φ_c in such a way that $c < d$ implies $\varphi_d\varphi_c^{-1}$ may be induced by a linear isometry $\psi_{d,c}$ of H_c in H_d which is unique up to multiplication by a number of modulus one. Our next task is to show that these arbitrary multipliers may be chosen consistently; i.e., so that

$$(15) \quad a < b < c \text{ implies } \psi_{c,a} = \psi_{c,b}\psi_{b,a} .$$

3. Erratum, page 1167, line 4 from bottom.

For " $\sum_{i=1}^n \lambda_i(u)\psi_{b_n, a_i}$ " read " $\sum_{i=1}^n \lambda_i(u)\psi_{b_n, a_i} u_i$ ".

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