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NEAL ZIERLER

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# ON THE LATTICE OF CLOSED SUBSPACES OF HILBERT SPACE

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**The purpose of this note is to answer two questions which have arisen in connection with the lattice-theoretic characterization of the set of closed subspaces of a Hilbert space of countably infinite dimension which appears in "Axioms for nonrelativistic quantum mechanics," Pacific Journal of Mathematics, Vol. 11, No. 3, 1961, pages 1151–1169.**

The material in this section is to replace [2, p. 1165, lines 10–32]. Up to that point it has been shown that the lattice under each finite element of $P$ is isomorphic to the lattice of subspaces of a Hilbert space over a field $D$ which is either real or complex. The orthocomplementation induced in a Hilbert space by such an isomorphism gives rise to an involution of $D$ (vide infra). In this section we show that such an involution is continuous, thereby closing a gap brought to our attention by a comment of M. D. Maclaren.

Let $a \in P_f$ with $n = \dim a > 0$. Choose pairwise orthogonal points $A_0, \cdots, A_n$ in $(a)$ and in each line $l_i = A_0 \vee A_i$, $i = 1, \cdots, n$, choose, a point $E_i$ different from $A_0$ and $A_i$. Clearly the points $A_0, E_1, \cdots, E_n$ are independent and the choice of $A_1 \vee \cdots \vee A_n$ as improper hyperplane, $A_0$ as origin and $E_1, \cdots, E_n$ as unit points leads to the unique introduction of homogeneous coordinates in $(a)$ in standard fashion. In particular, the proper points of $l_1$ are precisely those with homogeneous coordinates $(1, \lambda, 0, \cdots, 0)$ which we abbreviate as $(1, \lambda)$ — $\lambda$, of course, being any member of the field $D$ that has been constructed. The topology for $D$ is obtained as follows: The subset $N$ of $D$ is a neighborhood of 0 if $\{(1, \nu): \nu \in N\}$ is a neighborhood of $A_0$ in $l_1$. Under this topology, $D$ is either the real or complex field (cf. [2, Lemma 2.11 et seq., p. 1164]).

It is shown in [1] that there then exist an involution $\sigma$ of $D$ and numbers (= members of $D$) $\eta_0, \cdots, \eta_n$ such that

(1) $\eta_i^\sigma = \eta_i$,

(2) $\sum x_i \eta_i x_i^\sigma = 0$ if and only if all $x_i = 0$,

(3) If $(x_0, \cdots, x_n) \in (a)_0$, then $a(x_0, \cdots, x_n)'$ (the complement of $(x_0, \cdots, x_n)$ in $(a)$) $= \vee \{(y_0, \cdots, y_n) \in (a)_0 \colon \sum y_i \eta_i x_i^\sigma = 0\}$

Note that by (2), no $\eta_i$ is 0 and that $1, \eta_1/\eta_0, \cdots, \eta_n/\eta_0$ defines the same orthomorplementation as $\eta_0, \cdots, \eta_n$; i.e., we may assume that $\eta_0 = 1$.

Again confining our attention to $l_1$, observe that if $\lambda \neq 0$ and $l_1(1, \lambda)'$ (the point of $l_1$ orthogonal to the point $(1, \lambda)$) has coordinates

$(1, \mu)$, then $\mu = -1/\eta_1\lambda^\sigma$. Hence if $\lambda_m \to 1$ and is never 0, $(1, \lambda_m) \to (1, 1)$ by definition (of the topology for $D$) so $(1, \lambda_m)' \to (1, 1)' = (1, -1/\eta_1)$ by [2, Lemma 2.8]. But $(1, \lambda_m)' = (1, \mu_m)$ with $\mu_m = -1/\eta_1\lambda_m^\sigma$. Then $(1, \mu_m) \to (1, -1/\eta_1)$ which implies $\mu_m \to -1/\eta_1$; i.e., $-1/\eta_1\lambda_m^\sigma \to -1/\eta_1$ so $\lambda_m^\sigma \to 1$. Thus, $\sigma$ is continuous at 1 and hence is continuous (if $\lambda_m \to 0$ then $\lambda_m + 1 \to 1$ so $(\lambda_m + 1)^\sigma = \lambda_m^\sigma + 1 \to 1$ so $\lambda_m^\sigma \to 0$). Of course, this result was automatic in the real case. It follows that $\sigma$ is either the identity or, in the complex case, conjugation. It follows now from (2) that $\eta_1, \cdots, \eta_n$ are positive real numbers. If $D$ is the complex numbers, $\sigma$ is conjugation, for otherwise $(1, i\eta_1^{-1/2}, 0, \cdots, 0)$ would be self-orthogonal.

Taking the Hilbert space of $n + 1$ tuples of $D$ as $H_a$, the mapping $(x_0, \cdots, x_n) \to \{\lambda(x_0, \cdots, x_n): \lambda \in D\}$ clearly induces a continuous isomorphism $\varphi_a$ of $(a)$ on the lattice $L_a$ of subspaces of $H_a$ such that the orthocomplementation induced by $\varphi_a$ in $L_a$ is obtained from the inner product $(x, y) = \sum x_i\eta_i\bar{y}_i$ for $H_a$.

2. The following is a replacement for [2, p. 1165, lines 33 to 41]. Its purpose is to insure that all the isometries $\psi_{b,a}$ are linear rather than conjugate linear. I am indebted to V. S. Varadarajan for calling my attention to this omission.

Let $a \leqq b$ be finite and suppose that, in accordance with what has preceded, we have selected a Hilbert space $H_a$ over $D$ of dimension $1 + \dim a$ and a continuous isomorphism $\varphi_a$ of $(a)$ on the lattice $L_a$ of subspaces of $H_a$ which is orthogonality-preserving in the sense that

$$\varphi_a(c) \perp \varphi_a(d) \text{ if and only if } c \perp d . \tag{13}$$

Suppose that $H_b, \varphi_b$, have been similarly chosen for $b$.

Now $\varphi_b\varphi_a^{-1}$ is a continuous, orthogonality-preserving isomorphism of $L_a$ in $L_b$. Hence, as is well-known and not difficult to show, there exists a continuous automorphism $\sigma$ of $D$ and a $\sigma$-isometry $\psi_{b,a}$, unique up to multiplication by a number of modulus one, providing $\dim a > 0$ (see below), such that $\psi_{b,a}$ induces $\varphi_b\varphi_a^{-1}$ in the sense that $\varphi_b\varphi_a^{-1}[v] = [\psi_{b,a}v]$ for all $v \in H_a$, where $[v]$ denotes the linear subspace generated by $v$. A $\sigma$-isometry $\psi$ of $H$ is a mapping of $H$ in itself with the following three properties:

$$\begin{aligned} &\text{Additivity:} && \psi(u + v) = \psi(u) + \psi(v) \\ &\sigma\text{-linearty:} && \psi(\lambda u) = \lambda^\sigma\psi(u) \\ &\sigma\text{-isometry:} && (\psi(u), \psi(v)) = (u, v)^\sigma . \end{aligned} \tag{14}$$

A $\sigma$-isometry is said to be *linear* or *conjugate-linear* when $\sigma$ is the identity or conjugation respectively.

If $D$ is the real field, the automorphism $\sigma$ is the identity, while in the complex case, in view of its continuity, $\sigma$ may be either the identity or conjugation. Observe that if dim $a = 0$ and $u, v$ are unit vectors in $H_a$, $\varphi_b(a)$ respectively, then $\lambda u \to \lambda v$ and $\lambda u \to \bar{\lambda} v$ both induce the mapping $\varphi_b \varphi_a^{-1}$ of $L_a$ in $L_b$. In other words, $\psi_{b,a}$ may be chosen both linear and conjugate-linear when dim $a = 0$, independent of the choice of $H_a$, $\varphi_a$ and $H_b$, $\varphi_b$. In general, the linearity of $\psi_{b,a}$ may be achieved through the proper choice of $H_b$, $\varphi_b$ as follows. Suppose that $\psi_{b,a}$ inducing $\varphi_b \varphi_a^{-1}$ is conjugate-linear. Let $\{v_i\}$ be a complete orthonormal set for $H_b$ and define $\gamma: H_b \to H_b$ by: $\gamma(\sum \lambda_i v_i) = \sum \bar{\lambda}_i v_i$. Let $\varphi$ denote the automorphism of $L_b$ induced by $\gamma$ and let $\bar{\varphi}_b = \varphi \circ \varphi_b$. Then $\bar{\varphi}_b$ is a continuous, orthogonality-preserving isomorphism of $(b)$ on $L_b$ which is induced by the linear isometry $\bar{\psi}_{b,a} = \gamma \circ \psi_{b,a}$.

Suppose now that dim $a > 0$, that $H_a$, $\varphi_a$ have been chosen arbitrarily and that for every finite $b > a$, $H_b$, $\varphi_b$ has been chosen as above so that $\varphi_b \varphi_a^{-1}$ is "linear" in the sense that every isometry of $H_a$ in $H_b$ which induces it is linear. For each finite $c \not> a$ let $H_c = \varphi_{a \vee c}(c)$ and let $\varphi_c = \varphi_{a \vee c} \mid (c)$. Then $\varphi_{a \vee c} \varphi_c^{-1}$ is linear, for it is induced by the projection in $H_{a \vee c}$ of its subspace $H_c$.

Now that $H_c$, $\varphi_c$ have been assigned to every finite $c$, it remains to show that $\varphi_{c_1} \varphi_{c_2}^{-1}$ is in fact linear whenever $c_2 < c_1$. The type of argument we shall use involves the introduction of $c_3 < c_2$ for which both $\varphi_{c_1} \varphi_{c_3}^{-1}$ and $\varphi_{c_2} \varphi_{c_3}^{-1}$ are known to be linear. The linearity of $\varphi_{c_1} \varphi_{c_2}^{-1}$ then follows from the equation $\varphi_{c_1} \varphi_{c_3}^{-1} = (\varphi_{c_1} \varphi_{c_2}^{-1})(\varphi_{c_2} \varphi_{c_3}^{-1})$.

Given finite $c_2 < c_1$, let $b_i = c_i \vee a$, $i = 1, 2$. Now $b_2 \leqq b_1$ and $\varphi_{b_1} \varphi_{b_2}^{-1}$ is linear, for $\varphi_{b_i} \varphi_a^{-1}$, $i = 1, 2$ are linear by construction and $\varphi_{b_1} \varphi_a^{-1} = (\varphi_{b_1} \varphi_{b_2}^{-1})(\varphi_{b_2} \varphi_a^{-1})$. Since $\varphi_{b_1} \varphi_{b_2}^{-1}$ is linear and $\varphi_{b_2} \varphi_{c_2}^{-1}$ is linear by construction, $\varphi_{b_1} \varphi_{c_2}^{-1} = (\varphi_{b_1} \varphi_{b_2}^{-1})(\varphi_{b_2} \varphi_{c_2}^{-1})$ is linear. Finally, since $\varphi_{b_1} \varphi_{c_2}^{-1}$ is linear and $\varphi_{b_1} \varphi_{c_i}^{-1}$ is linearby construction, the linearity of $\varphi_{c_1} \varphi_{c_2}^{-1}$ follows from the equation $\varphi_{b_1} \varphi_{c_2}^{-I} = (\varphi_{b_1} \varphi_{c_1}^{-1})(\varphi_{c_1} \varphi_{c_2}^{-1})$.

Thus, each finite $c$ has been provided with $H_c$, $\varphi_c$ in such a way that $c < d$ implies $\varphi_d \varphi_c^{-1}$ may be induced by a linear isometry $\psi_{d,c}$ of $H_c$ in $H_d$ which is unique up to multiplication by a number of modulus one. Our next task is to show that these arbitrary multipliers may be chosen consistently; i.e., so that

$$a < b < c \text{ implies } \psi_{c,a} = \psi_{c,b} \psi_{b,a} . \tag{15}$$

3. Erratum, page 1167, line 4 from bottom.
For "$\sum_{i=1}^{n} \lambda_i(u) \psi_{b_n, a_i}$" read "$\sum_{i=1}^{n} \lambda_i(u) \psi_{b_n, a_i} u_i$".

## References

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2. N. Zierler, *Axioms for nonrelativistic quantum mechanics*, Pacific J. Math. **11** (1961), 1151-1169.

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