

# Pacific Journal of Mathematics

**ERRATUM: "UNIQUENESS AND EXISTENCE PROPERTIES OF  
BOUNDED OBSERVABLES"**

STANLEY P. GUDDER

should be replaced by  $\tilde{Z}$ , and  $\tilde{Z}$  should be replaced by  $\tilde{Z}^0$ . The symbols  $\tilde{\mathfrak{U}}_m$  and  $\tilde{\mathfrak{U}}_m^0$  should be replaced throughout by  $\mathfrak{U}_m^0$  and  $\mathfrak{U}_m^0$ , respectively; however,  $\tilde{\mathfrak{U}}_n$  and  $\tilde{\mathfrak{U}}_n^0$  remain unchanged. The first equation of line 14 page 235 should be ' $\mathfrak{U}_n = \tilde{\mathfrak{U}}_n$ .'

Correction to

## DUALITY AND TYPES OF COMPLETENESS IN LOCALLY CONVEX SPACES

WILLIAM B. JONES

Volume 18 (1966), 525-544

Proposition 2.14 is an obvious consequence of Lemma 2.8.

p. 538, line 5: The second equality is false in general for all  $\alpha$  (see [4]).

Some misprints:

- |        |  |
|--------|--|
| p. 526 | § 2 should start " $(\alpha, \beta) - \dots$ "             |
|        | line 3 of § 2, " $\alpha$ " instead of " $a$ "             |
| p. 528 | last line, remove final " $\}$ "                           |
| p. 532 | line 14, second " $\varepsilon$ " should be " $\epsilon$ " |
| p. 535 | line 2, should read  |
|        | $\dots \leq \frac{\varepsilon}{r} (r - \dots$              |
| p. 537 | line 8, second " $=$ " should be " $-$ "                   |
| p. 541 | line 9, " $\lambda_0$ " instead of " $1_0$ "               |

Correction to

## UNIQUENESS AND EXISTENCE PROPERTIES OF BOUNDED OBSERVABLES

S. P. GUDDER

Volume 19 (1966), 81-93

The author recently discovered that the proof of the corollary to Theorem 4.5 is incorrect, thus invalidating Theorem 4.6. We show now that Theorem 4.6 is still true for a class of observables with infinite spectra and prove a generalization of Theorem 4.5.

An observable  $x$  is *semi-bounded above (below)* if there is a number

$-\infty < c < \infty$  such that  $\sigma(x) \subset \{\lambda : \lambda \leq c\}$  ( $\sigma(x) \subset \{\lambda : \lambda \geq c\}$ ). The following not only generalizes Theorem 4.5 but gives a much simpler proof.

**THEOREM 1.1.** *Let  $x$  and  $y$  be observables on a quite full logic which are semi-bounded above and suppose that  $m(x)$  exists if and only if  $m(y)$  exists and in that case  $m(x) = m(y)$ . Then  $\lambda_0 = \max \{\lambda : \lambda \in \sigma(x)\} = \max \{\lambda : \lambda \in \sigma(y)\}$  and  $x(\lambda_0) = y(\lambda_0)$ .*

*Proof.* The first part of the conclusion follows just as in Theorem 4.5. Now suppose  $m[x(\lambda_0)] = 1$ , and  $m[y(\lambda_0)] \neq 1$ . Then there is a number  $\mu < \lambda_0$  such that  $m[y(-\infty, \mu)] > 0$ . Now since  $m(x)$  exists, so does  $m(y)$  and we have

$$\begin{aligned} \lambda_0 = m(x) = m(y) &= \int_{(-\infty, \lambda_0]} \lambda m[y(d\lambda)] = \left( \int_{(-\infty, \mu)} + \int_{[\mu, \lambda_0]} \right) \lambda m[y(d\lambda)] \\ &\leq \mu m[y(-\infty, \mu)] + \lambda_0 m[y[\mu, \lambda_0]] < \lambda_0 . \end{aligned}$$

which is a contradiction. Thus  $m[y(\lambda_0)] = 1$  whenever  $m[x(\lambda_0)] = 1$  and hence  $x(\lambda_0) \leq y(\lambda_0)$ . By symmetry  $x(\lambda_0) = y(\lambda_0)$ .

Of course the same result holds for observables which are semi-bounded from below.

**THEOREM 1.2.** *Let  $x$  and  $y$  be bounded observables on a quite full logic and suppose the spectrum of  $x$  has at most one limit point. If  $m(x) = m(y)$  for all  $m \in M$  then  $x = y$ .*

*Proof.* The most general such  $x$  has a point  $\lambda_0 \in \sigma(x)$  which is a limit point from both above and below of elements of  $\sigma(x)$ . The other cases will follow in a similar manner. We can assume without loss of generality that  $\lambda_0 = 0$ . Let the points of  $\sigma(x)$  be ordered as follows:  $\mu_1 < \mu_2 < \dots < \lambda_0 < \dots < \lambda_2 < \lambda_1$ . Now by Theorem 1.1  $\max \{\lambda : \lambda \in \sigma(y)\} = \lambda_1$  and  $y(\lambda_1) = x(\lambda_1)$ . Now let  $x_1 = x - \lambda_1 \chi_{\lambda_1}(x)$  and let  $y_1 = y - \lambda_1 \chi_{\lambda_1}(y)$ . Letting  $f$  be the identity function  $f(\lambda) = \lambda$  we have for  $E \in B(R)$

$$\begin{aligned} x_1(E) &= (f - \lambda_1 \chi_{\lambda_1})(x)(E) = x[(f - \lambda_1 \chi_{\lambda_1})^{-1}(E)] \\ &= \begin{cases} x(E) \wedge x(\lambda_1)' & \text{if } 0 \in E \\ x(E) \vee x(\lambda_1) & \text{if } 0 \notin E \end{cases} \dots\dots (1) . \end{aligned}$$

It is now easy to see that

$$\sigma(x_1) = \sigma(x) \cap \{\lambda_1\}' ; x_1(\lambda_i) = x(\lambda_i), i = 2, 3, \dots ;$$

and

$$x_1(\mu_i) = x(\mu_i), i = 1, 2, \dots .$$

Now

$$m(x_i) = m(x) - \lambda_1 m[x(\lambda_1)] = m(y) - \lambda_1 m[y(\lambda_1)] = m(y_1) .$$

Applying Theorem 1.1,  $\lambda_2 = \max \{\lambda : \lambda \in \sigma(y_1)\}$  and  $y_1(\lambda_2) = x_1(\lambda_2) = x(\lambda_2)$ . It now follows by applying (1) to  $y_1$  and  $y$  that  $\lambda_2$  is the second largest number in  $\sigma(y)$  and  $y(\lambda_2) = y_1(\lambda_2) = x(\lambda_2)$ . Continuing this process with the  $\lambda_i$ 's and also the  $\mu_i$ 's we have  $\{\lambda_i, \mu_i : i = 1, 2, \dots\} \subset \sigma(y)$  and  $y(\lambda_i) = x(\lambda_i)$ ,  $y(\mu_i) = x(\mu_i)$ ,  $i = 1, 2, \dots$ . Since  $\lambda_0$  is a limit point of the  $\lambda_i$ 's it follows that  $\lambda_0 \in \sigma(y)$ ,  $\{\lambda_i, \mu_i : i = 1, 2, \dots\} = \sigma(y)$  and

$$\begin{aligned} y(\lambda_0) &= y(\{\lambda_i, \mu_i : i = 1, 2, \dots\}') = [\Sigma y(\lambda_i) + \Sigma y(\mu_i)]' \\ &= [\Sigma x(\lambda_i) + \Sigma x(\mu_i)]' = x(\lambda_0) . \end{aligned}$$

Hence  $y = x$ .

A similar technique may be used to prove:

**COROLLARY 1.3.** *Let  $x$  and  $y$  be observables on a quite full logic which are semi-bounded from above (below) and suppose the spectrum of  $x$  has no finite limit point (this includes the possibility of a limit point at  $-\infty(+\infty)$ ). Suppose  $m(y)$  exists if and only if  $m(x)$  exists and in that case  $m(y) = m(x)$ . Then  $x = y$ .*

We close with a slightly strengthened form of Lemma 6.2 [1].

**LEMMA 1.4.** *If  $L$  is quite full and has Property E, then  $L$  is a lattice and  $m(a) = m(b) = 1$  implies  $m(a \wedge b) = 1$ .*

*Proof.* That  $L$  is a lattice follows from Lemma 6.2 [1]. If  $m(a) = m(b) = 1$ , then  $m(x_a + x_b) = m(a) + m(b) = 2$  and hence  $1 = m[(x_a + x_b)\{2\}] = m(a \wedge b)$ .

This last lemma is of interest since it rules out the counter-example of Section 5 [1] and is thus a possible sufficient condition for Property E.

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