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A REPRESENTATION THEOREM FOR ABELIAN GROUPS WITH NO ELEMENTS OF INFINITE P-HEIGHT

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The purpose of this note is to give a generalization of the representation Theorems 33.1 and 33.2 of [2]. Let G be an arbitrary abelian group and $B = [\bigoplus_{\lambda \in \Lambda} \langle x_\lambda \rangle] \oplus [\bigoplus_{i \geq 1} B_i]$ be a p-basic subgroup of G, cf. [3], where $\bigoplus_{\lambda \in \Lambda} \langle x_\lambda \rangle$ is the torsion-free part. For all $\lambda \in \Lambda$ let $(F_p^*)_\lambda$ be a copy of the group of p-adic integers, and let $(F_p)_\lambda$ denote the infinite cyclic group of finite p-adic integers in $(F_p^*)_\lambda$. Then G can be mapped homomorphically into the complete direct sum $[\bigoplus^*_{\lambda \in \Lambda} (F_p^*)_\lambda] \oplus [\bigoplus^*_{i \geq 1} B_i]$ with kernel $p^\omega G$. Furthermore, the image of G is a p-pure subgroup which contains $[\bigoplus_{\lambda \in \Lambda} (F_p)_\lambda] \oplus [\bigoplus_{i \geq 1} B_i]$ as a p-basic subgroup and is in turn contained in the p-adic completion of this subgroup (See Section 1 for definitions). This representation is completely analogous to the representation theorem for p-groups which is contained as a special case, and hopefully it is of similar use.

Definitions and facts concerning p-adic and n-adic topologies. In this article we list the definitions and facts concerning p-adic and n-adic topologies that are needed in this paper. For references see [2], [3], and [5].

DEFINITION 1.1. The p-adic topology for an abelian group is the topology with the subgroups $p^n G$, $n = 1, 2, \cdots$ as a basis for the neighborhoods of 0.

DEFINITION 1.2. The n-adic topology for an abelian group G is the topology with the subgroups $n!G$, $n = 1, 2, \cdots$ as a basis for the neighborhoods of 0.

DEFINITION 1.3. The completion of an abelian group in the p-adic (resp. n-adic) topology is its metric space completion with respect to the metric $d(x, y) = 10^{-m}$, where m is the largest integer such that $x - y \varepsilon p^m G$(resp. $m!G$).

PROPOSITION 1.4. If H is a p-pure (resp. pure) subgroup of the abelian group G, then the p-adic (resp. n-adic) topology of the subgroup is the same as the induced p-adic (resp. n-adic) topology.

THEOREM 1.5. *If an abelian group is complete in the n-adic topology, then it is a direct summand of every abelian group that* *contains it as a pure subgroup.*

PROPOSITION 1.6. A subgroup H of an abelian group G is dense in the p-adic (resp. n-adic) topology if and only if the quotient group G/H is p-divisible (respectively divisible).

2. The representation theorems. Let G be an abelian group, let B be a p-basic subgroup, cf. Fuchs [3], of G, and we write $B = \bigoplus_{n \geqq 0} B_n$ and $B_0 = \bigoplus_{\lambda \in \Lambda} \langle x_\lambda \rangle$. As in [1 p. 325], for each $g \in G$ and each natural n, we can write

2.1. $g = b_0^{(n)} + b_1 + \cdots + b_n + b_n^* + p^n g_n$ where $b_0^{(n)} \in B_0$, $b_i \in B_i$ for $1 \leqq i \leqq n$, $b_n^* \in \bigoplus_{i>n} B_i$, and $g_n \in G$. It is proved in [1] p. 326 that the b_i, $i \geqq 1$, are unique in any such representation, and that, given two such representations, one for n and one for m, we have

2.2 $$b_0^{(n)} - b_0^{(m)} \in p^{\min(m,n)} G\,.$$

For each λ, let $(F_p^*)_\lambda$ be the group of p-adic integers, and let $(F_p)_\lambda$ be the infinite cyclic subgroup of finite p-adic integers. We introduce the notation $P_1 = \bigoplus^*_{\lambda \in \Lambda} (F_p^*)_\lambda$, and $P_2 = \bigoplus^*_{i \geqq 1} B_i$. P_1 and P_2 are complete groups in the n-adic topology, and the n-adic topology coincides with the p-adic topology. $\bigoplus_{\lambda \in \Lambda} (F_p^*)_\lambda$ and $\bigoplus_{i \geqq 1} B_i$ are pure subgroups of P_1 and P_2, hence they possess completions in P_1 and P_2 for the coinciding n-adic and p-adic topologies. Let $C_1 = [\bigoplus_{\lambda \in \Lambda} (F_p^*)_\lambda]^*$ and $C_2 = [\bigoplus_{i \geqq 1} B_i]^*$, where the *indicates the completion. Notice that C_i is a direct summand of P_i, $i = 1, 2$.

We define a map $\sigma : G \to P_1 \bigoplus P_2$ as follows. Let g have the representation 2.1 for each n. Write $b_0^{(n)} = \sum_{\lambda \in \Lambda} m_\lambda^{(n)} x_\lambda$, and write $m_\lambda^{(n)}$ in its p-adic expansion

2.3 $$m_\lambda^{(n)} = \sum_{k \geqq 0} a_{\lambda,k}^{(n)} p^k,\ 0 \leqq a_{\lambda,k}^{(n)} \leqq p-1\,.$$

It follows from 2.2 that $a_{\lambda,k}^{(n)}$ is independent of n for $k < n$. Now define

2.4 $$g\sigma = (\cdots, \sum_{k \geqq 0} a_{\lambda,k}^{(k+1)} p^k, \cdots; b_1, b_2, \cdots)\,.$$

THEOREM 2.5. *The map σ is a homomorphism, and ker $\sigma = p^\omega G$, the subgroup of elements of infinite p-height. The p-basic subgroup B of G is mapped onto the group $[\bigoplus_{\lambda \in \Lambda} (F_p)_\lambda] \bigoplus [\bigoplus_{i \geqq 1} B_i]$ which is a p-basic subgroup of $C_1 \bigoplus C_2$.*

Proof. It is easy to see that σ is a homomorphism. Let $g \in p^\omega G$, and write g as in 2.1. Then by the p-purity of B, each of $b_0^{(n)}$, $b_1 \cdots, b_n, b_n^*$ is divisible by p^n in the summand of B to which it belongs. Hence $b_1 = \cdots = b_n = 0$. Since $b_0^{(n)}$ is divisible by p^n in in B_0, it follows that in $m_\lambda^{(n)} = \sum_{k\geqq 0} a_{\lambda,k}^{(n)} p^k$ the coefficient $a_{\lambda,k}^{(n)} = 0$ for $k \leqq n-1$. Thus $g\sigma = 0$. Conversely, assume $g\sigma = 0$. Then in the representation 2.1, $b_1 = b_2 = \cdots = b_n = 0$, and in the equation $m_\lambda^{(n)} = = \sum_{k\geqq 0} a_{\lambda,k}^{(n)} p^k$, $0 \leqq a_{\lambda,k}^{(n)} \leqq p-1$, we have $a_{\lambda,k}^{(k+1)} = 0$ for each k. The uniqueness of the $a_{\lambda,k}^{(n)}$ for $k < n$ implies $a_{\lambda,k}^{(n)} = 0$ for $0 \leqq k < n$, i.e. $m_\lambda^{(n)}$ is divisible by p^n. Thus $b_0^{(n)}$ is divisible in B_0 by p^n. The remainder of this part of the proof is exactly as in the proof of Theorem 3 in [1] pp. 326-7. It is obvious from 2.1 that B is mapped onto

$$[\bigoplus_{\lambda\in\Lambda} (F_p)_\lambda] \oplus [\bigoplus_{i\geqq 1} B_i] ,$$

and it is easy to check that this is a p-basic subgroup of $C_1 \oplus C_2$

THEOREM 2.6. *$G\sigma$ is p-pure in $P_1 \oplus P_2$, and $(G\sigma)^* = C_1 \oplus C_2$, where *indicates the completion in the p-adic topology.*

Proof. By 2.5 $B\sigma$ is a p-pure subgroup of $P_1 \oplus P_2$. Since $G\sigma/B\sigma$ is a p-divisible (hence p-pure) subgroup of $(P_1 \oplus P_2)/B\sigma$, it follows that $G\sigma$ is a p-pure subgroup of $P_1 \oplus P_2$. Since $G\sigma$ is p-pure in $P_1 \oplus P_2$ it possesses a p-adic completion in $P_1 \oplus P_2$. $B\sigma \leqq G\sigma$ implies $C_1 \oplus C_2 = (B\sigma)^* \leqq (G\sigma)^*$, and since $B\sigma$ is dense in $G\sigma$ in the p-adic topology, $G\sigma \leqq (B\sigma)^* = C_1 \oplus C_2$, thus $(G\sigma)^* \leqq C_1 \oplus C_2$.

COROLLARY 2.7. *Every abelian group G with no elements of infinite p-height may be considered to be a p-pure subgroup of some group $[\bigoplus^*_{\lambda\in\Lambda} (F_p^*)_\lambda] \oplus [\bigoplus^*_{i\geqq 1} B_i]$ and containing $[\bigoplus_{\lambda\in\Lambda}(F_p)_\lambda] \oplus [\bigoplus_{i\geqq 1} B_i]$ as a p-basic subgroup.*

If G is a p-group, then $P_1 = 0$, and $G\sigma \leqq (C_2)_t$, the torsion subgroup of C_2. Thus in this case our theorems are exactly the important and useful Theorems 33.1 and 33.2 of [2].

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