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**A TOPOLOGICAL CHARACTERIZATION OF GLEASON PARTS**

JOHN BRADY GARNETT

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**Let  $A$  be a function algebra on its maximal ideal space  $M(A)$ , and let  $P$  be a Gleason part of  $M(A)$ . It is easily seen that  $P$  is then a  $\sigma$ -compact completely regular space. We prove the converse: if  $K$  is completely regular and  $\sigma$ -compact, then there exists a function algebra whose maximal ideal space contains a part homeomorphic to  $K$ . Every bounded continuous function on that part is the restriction of a function in the given algebra. Consequently no subset of the part can have an analytic structure.**

Suppose  $X$  is a compact Hausdorff space and  $A$  is a subalgebra of  $C(X)$ , the algebra of continuous complex valued functions on  $X$ . Assume  $A$  separates the points of  $X$ , contains the constant functions, and is uniformly closed.  $A$  is then called a function algebra on  $X$ . With the weak star topology, the maximal ideal space  $M(A)$  of  $A$  is a compact Hausdorff space. We consider  $X$  as embedded in  $M(A)$  and  $A$  as a function algebra on  $M(A)$ .

In [4] Gleason noted that an equivalence relation could be defined on  $M(A)$  by setting  $x \sim y$  when the functional norm  $\|x - y\|_{A^*} < 2$ . The equivalence classes for this relation are called the "parts" of  $M(A)$ . In certain cases parts have been used to impose an analytic structure on  $M(A)$  (see for example [7]).

Let  $P$  be a part of some  $M(A)$ . Then clearly  $P$  is a completely regular space and fixing  $p \in P$  we have

$$P = \bigcup_{n=1}^{\infty} \{q \in M(A) : \|p - q\| \leq 2 - 1/n\},$$

where each term in the union is weak star closed, and hence compact, so that  $P$  is  $\sigma$ -compact.

Some results in this paper have been announced in [3].

2. We begin with a theorem which will be our basic tool in constructing parts.

**THEOREM 1.** *Let  $A$  be a function algebra,  $S$  a hull-kernel closed subset of  $M(A)$  and  $P$  a part of  $M(A)$ . Then there is a function algebra  $B$  such that  $M(B)$  contains a part  $Q$  homeomorphic to  $P \cap S$ . Moreover,  $B|_Q$  is isometrically isomorphic to  $A|_{P \cap S}$ .*

Let  $\alpha$  be a positive irrational number, and denote by  $A_\alpha$  the function algebra on the torus  $T^2$  generated by the functions  $z \rightarrow z_1^m z_2^n$  where  $m + n\alpha \geq 0$ . Let  $m^0$  be the point in  $M(A_\alpha)$  represented by Haar measure on  $T^2$  (which is multiplicative on  $A_\alpha$ ). Then  $m^0 \notin T^2$  and  $\{m^0\}$  is a part of  $M(A_\alpha)$  ([5] p. 316). If  $J$  is a proper closed subset of  $T^2$ , then  $A_\alpha|_J$  is dense in  $C(J)$  ([8] pp. 69-70), so that when  $x \in M(A_\alpha) \setminus J$  there is a function  $f \in A_\alpha$  such that  $|f(x)| > \max_{z \in J} |f(z)|$ , as otherwise evaluation at  $x$  would induce a complex homomorphism of  $C(J)$ .

*Proof of Theorem 1.* Let  $A_\alpha \otimes A$  be the function algebra on  $M(A_\alpha) \times M(A)$  generated by the functions of the form  $(x, y) \rightarrow f(x)g(y)$  where  $f \in A_\alpha$  and  $g \in A$ .  $M(A_\alpha \otimes A)$  is homeomorphic to  $M(A_\alpha) \times M(A)$  in a natural fashion.

Set  $J = \{z \in T^2 : \text{Real } z_1 \leq 0\}$  and

$$X = (J \times M(A)) \cup (M(A_\alpha) \times S).$$

$X$  is a compact subset of  $M(A_\alpha) \times M(A)$ . Our algebra  $B$  is the uniform closure on  $X$  of  $\{h|_X : h \in A_\alpha \otimes A\}$ .  $M(B)$  is then the  $A_\alpha \otimes A$ -hull of  $X$ ,

$$\{q \in M(A_\alpha \otimes A) : |g(q)| \leq \max_{p \in X} |g(p)| \text{ for all } g \in A_\alpha \otimes A\}.$$

If  $(x^0, y^0) \in M(A_\alpha \otimes A) \setminus X$ , then  $x^0 \notin J$  and  $y^0 \notin S$ . As  $S$  is hull-kernel closed, there is a function  $g$  in  $A$  with  $g(y^0) = 1$  and  $g(S) = 0$ . As  $x^0 \notin J$ , there is a function  $f$  in  $A_\alpha$  with  $f(x^0) = 1$  and  $\max_{z \in J} |f(z)| < 1$ . Replace  $f$  by a suitable power  $f^n$  so that  $\max_{z \in J} |f^n(z)| < (1/\|g\|)$ . Then  $h(x, y) = f^n(x)g(y)$  is in  $A_\alpha \otimes A$  and  $h(x^0, y^0) = 1$  while  $\max_{p \in X} |h(p)| < 1$ . Hence  $M(B) = X$ .

Take  $Q = \{m^0\} \times (P \cap S)$ . Then  $Q$  is subset of  $X$ . For  $s \in P \cap S$ , let  $p_s = (m^0, s) \in Q$ . Let  $(x^0, y^0) \in X \setminus Q$ . If  $x^0 \neq m^0$ , then using functions of the variable  $x \in M(A_\alpha)$  alone we see that  $(x^0, y^0) \not\sim p_s$  for any  $s \in P \cap S$ . Similarly if  $y^0 \notin P$ , then  $(x^0, y^0) \not\sim p_s$  for all such  $s$ . Finally if  $y^0 \in S$ , then by the choice of  $X$ ,  $x^0 \neq m^0$ . Hence  $Q$  is a union of parts.

If  $s \in P \cap S$  and  $g \in B$ , then

$$g(p_s) = \int_{T^2} g(x, s) d\lambda$$

where  $\lambda$  is normalized Haar measure on  $T^2$ , because  $\lambda$  represents  $m^0$  for  $A_\alpha$ . Take  $s$  and  $t$  in  $P \cap S$ ,  $g \in B$  with  $\|g\| \leq 1$ . Then

$$\begin{aligned} |g(p_s) - g(p_t)| &\leq \int_{T^2} |g(x, s) - g(x, t)| d\lambda \\ &\leq \int_{T^2 \setminus J} 2d\lambda + \int_J |g(x, s) - g(x, t)| d\lambda. \end{aligned}$$

Now if  $x \in J$ , then  $\{x\} \times M(A) \subset X$  so that  $y \rightarrow g(x, y)$  is in  $A$  with norm  $\leq 1$ . Therefore there is a constant  $c < 2$  such that for each  $x \in J$   $|g(x, s) - g(x, t)| < c$ , because  $s \sim t$ . Hence

$$|g(p_s) - g(p_t)| \leq \int_{x^2, J} 2d\lambda + \int_J cd\lambda < 2$$

and  $p_s \sim p_t$ . Thus  $Q$  is a part.

It is obvious that  $Q$  is homeomorphic to  $P \cap S$  and that  $B|Q = A|P \cap S$ , because the coordinate  $x$  is constant on  $Q$ .

As a corollary to Theorem 1 we now prove a special case of our main result because in this case the proof is much simpler.

$D$  denotes the closed unit disc in the complex plane and  $D^\circ$  its interior.  $A_0$  is the algebra of all functions continuous on  $D$  and analytic on  $D^\circ$ . If  $K$  is a locally compact Hausdorff space, then  $K^* = K \cup \{\infty\}$  is its one point compactification.

**COROLLARY.** *Let  $K$  be a locally compact  $\sigma$ -compact Hausdorff space. Then there exists a function algebra  $B$  such that  $M(B)$  contains a part  $Q$  homeomorphic to  $K$ . Moreover  $B|Q$  is isometrically isomorphic to  $C(K^*)|K$ .*

*Proof.* Let  $A = \{f \in C(K^* \times D) : f| \{x\} \times D \in A_0 \text{ for each } x \in K^* \text{ and } f| K^* \times \{0\} \text{ is constant}\}$ . Then  $M(A) = K^* \times D / \approx$  where  $\approx$  identifies  $K^* \times \{0\}$  to a point, and  $P = \{(x, z) \in M(A) : |z| < 1\}$  is a part in  $M(A)$ , as  $P$  is a union of discs with the centers identified.

Since  $K$  is  $\sigma$ -compact,  $\{\infty\}$  is a  $G_\delta$  set in  $K^*$ . Hence there is a continuous function  $h: K^* \rightarrow [1/2, 1]$  such that  $h^{-1}(1) = \{\infty\}$ . Let  $S \subset M(A)$  be the graph of  $h$ ,  $S = \{(x, h(x)) : x \in K^*\}$ . Then the function  $g(x, z) = (h(x) - z/3h(x) - z)$  is in  $A$  and vanishes exactly on  $S$ , so that  $S$  is hull-kernal closed. And clearly  $S \cap P$  is homeomorphic to  $K$ . Finally if  $f \in C(K^*)$ , then  $f'(x, z) = zf(x)/h(x)$  is in  $A$  and  $f'(x, h(x)) = f(x)$  when  $x \in K$ . Thus  $A|S \cap P \cong C(K^*)|K$ . The conclusion of the corollary now follows directly from Theorem 1.

3. Before proving our main theorem we construct the algebra to be used in place of the disc algebra  $A_0$ . Let  $I$  be an index set, and let  $Y_I$  be the product of discs,  $Y_I = \prod(D : i \in I)$ . Denote by  $A_I$  the subalgebra of  $C(Y_I)$  generated by the coordinate functions  $z_i, i \in I$  where  $z_i(p) = p_i$ . Then  $M(A_I) = Y_I$ , for if  $\varphi \in M(A_I)$ , then  $|\varphi(z_i)| \leq 1$  so that  $\varphi$  is evaluation at  $\lambda \in Y_I$  where  $\lambda_i = \varphi(z_i)$ . Let  $\theta$  be the "origin" in  $Y_I$ ,  $z_i(0) = 0$  for all  $i \in I$ , and let  $P_0$  be the part of  $M(A_I)$  containing  $\theta$ . We now need a well known fact which is proved using elementary conformal mappings of the disc  $D$ . If  $(g_n)_{n=1}^\infty$  is a sequence in  $A$  with  $\|g_n\| \leq 1$  and  $g_n(x) \rightarrow 1$ , then  $x \sim y$  implies  $g_n(y) \rightarrow 1$ .

LEMMA. Let  $p \in M(A_I)$ . Then  $p \in P_0$  if and only if there exists  $a < 1$  such that  $|z_i(p)| \leq a$  for all  $i \in I$ .

*Proof.* If no such number exists, then there is a sequence  $(g_n)_{n=1}^{\infty}$  of coordinate functions  $z_i$  such that  $g_n(p) \rightarrow 1$  while  $g_n(\theta) = 0$ . Hence  $p \not\sim \theta$ , by the above remark. If such an  $a$  exists, then let  $\rho: D \rightarrow Y_I$  by  $(\rho(t))_i = t/a \cdot p_i$ . Then  $\rho(0) = \theta$ ,  $\rho(a) = p$  and for  $f \in A_I$ ,  $f \circ \rho \in A_0$  with  $\|f \circ \rho\| \leq \|f\|$ . Then as  $0 \sim a$  for  $A_0$  we have  $0 \sim p$ .

THEOREM 2. Let  $K$  be a  $\sigma$ -compact completely regular space. Then there is an algebra  $B$  and a part  $Q \subset M(B)$  such that  $Q$  is homeomorphic to  $K$  and  $B|Q \cong C^b(K)$ , the algebra of bounded continuous functions on  $K$ .

*Proof.* Let  $\beta K$  be the Stone-Ćech compactification of  $K$ . Take<sup>(1)</sup>  $I = \beta K \setminus K$  and set  $A = \{f \in C(\beta K \times Y_I) : f|_{\{x\} \times Y_I} \in A_I \text{ for all } x \in \beta K \text{ and } f|_{\beta K \times \{\theta\}} \text{ is constant}\}$ . Then  $M(A) = \beta K \times Y_I / \approx$  where  $\approx$  identifies  $\beta K \times \{\theta\}$  to a point, and  $P = \{(x, z) \in M(A) : z \in P_0\}$  is a part of  $M(A)$ .

Write  $K = \bigcup_{n=1}^{\infty} K_n$ , where  $K_n \subset K_{n+1}$ , and each  $K_n$  is compact. Then for each  $t \in \beta K \setminus K$  there exists a continuous function  $h_t: \beta K \rightarrow [1/2, 1]$  with  $h_t(t) = 1$  and  $h_t(x) \leq 1 - 2^{-n}$  when  $x \in K_n$ . Let  $\rho: \beta K \rightarrow M(A)$  be defined by  $\rho(x) = (x, H(x))$  where  $(H(x))_t = h_t(x)$  for each  $t \in \beta K \setminus K$ . Then  $\rho$  is a homeomorphism of  $\beta K$  onto  $S = \rho(\beta K)$  and  $\rho(K) = S \cap P$  by the above lemma.  $S$  is hull kernel closed in  $M(A)$  because  $S = \bigcap \{g_t^{-1}(0) : t \in \beta K \setminus K\}$  where  $g_t(x, z) = (h_t(x) - z_t)/3h_t(x) - z_t$ . And  $A|S \cap P \cong C^b(K)$ , because if  $f \in C^b(K)$  and  $\tilde{f}$  is its unique extension to  $\beta K$ , then for any  $t \in \beta K \setminus K$ ,  $f'(x, z) = z_t \tilde{f}(x)/h_t(x)$  is in  $A$  and  $f' = \tilde{f} \circ \rho^{-1}$  on  $S$ . The conclusion now follows from Theorem 1.

We remark that with these arguments one can get some restriction algebras  $B|Q$  different from  $C^b(K)$ . For example, if  $K$  is compact and  $A_1$  is an algebra with  $M(A_1) = K$ , then there is an algebra  $B$  with part  $Q$  homeomorphic to  $K$  and  $B|Q = A_1$ .

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<sup>1</sup> If  $K$  is compact, let  $I$  be a singleton, and proceed as in the corollary.

<sup>2</sup> She observed that  $X = \{(z, w) : |z| \leq 1, w = \pm 1/2\} \cup \{(z, w) : |z| = 1, \operatorname{Im} z \geq 0, |w| \leq 1\}$  is a polynomially convex subset of  $\mathbb{C}^2$  containing a part consisting of two discs.

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