SOME METRICAL THEOREMS IN NUMBER THEORY

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In this paper some metrical theorems on Diophantine approximation, continued fractions and θ-adic expansions are proved.

In the first part some of the common properties of the following transformations from the unit interval onto itself are investigated. Denote by \( \{a\} \) the fractional part of \( a \),

A. \( T: a \to \{aa\} \quad a > 1 \) integer

which describes the expansion of \( a \) in the scale \( a \)

B. \( T: a \to \{\frac{1}{a}\} \)

which describes the continued fractions

C. \( T: a \to \{\theta a\} \quad \theta > 1 \) noninteger

which describes the expansion of \( a \) as a \( \theta \)-adic fraction.

The main theorem of the first part (Theorem 2) gives an estimate of the number of solutions of the system of inequalities

\[ T^k a \in I_k \quad 1 \leq k \leq n \]

where \( n \) is an integer, \( T \) is any of these three transformations and \( (I_k) \) is an arbitrary sequence of intervals contained in the unit interval.

It generalises and refines well known theorems on the distribution function of the sequence \((T^k a)\). Theorem 2 follows from a very general theorem—a quantitative Borel-Cantelli Lemma.

It is also shown that \( T \) is strongly mixing (Theorem 1). The second part of the paper deals with the metric theory of continued fractions. Theorems of LeVeque and Bernstein are refined.

1. Frequently a real number \( \alpha \) is represented in one of the following ways:

A. in the scale \( a \), where \( a > 1 \) is an integer,

B. as a continued fraction,

C. as a \( \theta \)-adic fraction, where \( \theta > 1 \) is a noninteger.

Let us recall some of the properties of these representations:

A. If \( a > 1 \) denotes an integer then every \( \alpha \in [0, 1) \) can be written as

\[ \alpha = \sum_{k=1}^{\infty} \frac{c_k}{a^k} = \sum_{k=1}^{n} \frac{c_k}{a^k} + \frac{y_{n+1}}{a^n} \]
where the digits $c_k$ are nonnegative integers less than $a$ and $0 \leq y_n < 1$.
The representation can be made unique.

Define on $[0, 1)$ the transformation $T$

$$T: \alpha \rightarrow \{a\alpha\}$$

Clearly for $n \geq 0$ we have

$$y_{n+1} = y_{n+1}(\alpha) = T^*\alpha = \{a^n\alpha\}.$$ 

As is well known $T$ preserves the Lebesgue-measure and $T$ is ergodic.
(For definitions and theorems in ergodic theory see Halmos [5] pp. 5–37).

B. Every $\alpha \in (0, 1]$ can be expressed as a simple continued fraction

$$(1) \alpha = [a_1, a_2, \cdots]$$

where the partial quotients $a_i = a_i(\alpha)$ are positive integers. Again,
the representation can be made unique. If $\alpha$ is given by (1) then
the finite continued fraction

$$[a_1, a_2, \cdots, a_n] = \frac{p_n}{q_n}$$

is called the $n$-th convergent. It is in its lowest terms.

Define on $(0, 1]$ the transformation $T$

$$T: \alpha \rightarrow \left\{ \frac{1}{\alpha} \right\}$$

or else

$$T([a_1(\alpha), a_2(\alpha), \cdots]) = [a_2(\alpha), a_3(\alpha), \cdots].$$

Then clearly for $n \geq 0$ we have

$$a_{n+1}(\alpha) = a_i(T^n\alpha)$$

and

$$z_{n+1}(\alpha) = T^n\alpha = [a_{n+1}(\alpha), a_{n+2}(\alpha), \cdots].$$

Define on $(0, 1]$ a measure $\mu$ by setting

$$\mu(E) = \frac{1}{\log 2} \int_E \frac{dx}{1 + x}$$

for every Lebesgue-measurable set $E$. Knopp [13] proved that $T$ is
ergodic with respect to the Lebesgue-measure. Ryll-Nardzewski [21]
showed that $T$ preserves $\mu$ and that $\mu$ is equivalent to the Lebesgue-
measure (i.e. both are absolutely continuous to each other).

C. For a fixed noninteger $\theta > 1$ define the transformation $T$ on $[0, 1)$

\[ T: \alpha \rightarrow \{\theta \alpha\}. \]

Put

\[ x_1 = \alpha, \quad x_2 = \{\theta x_1\}, \ldots, \quad x_{n+1} = \{\theta x_n\}, \ldots \]

\[ \lambda_1 = [\theta \alpha], \quad \ldots, \quad \lambda_n = [\theta x_n], \ldots \]

For $n \geq 0$ we have $x_{n+1} = x_{n+1}(\alpha) = T^n \alpha$. Clearly $0 \leq \lambda_n < \theta$, $0 \leq x_n < 1$ and

\[ \alpha = \sum_{k=1}^{\infty} \frac{\lambda_k}{\theta^k} = \sum_{k=1}^{n} \frac{\lambda_k}{\theta^k} + \frac{x_{n+1}}{\theta^n}. \]

Rényi [19] proved that there exists a unique measure $\mu$ invariant under $T$ and equivalent to the Lebesgue measure and furthermore that $T$ is ergodic with respect to $\mu$.

Write $\lambda_0(1) = \lambda$ and $x_0(1) = t$. Denote by $\varphi_\gamma(t)$ the characteristic function of $[0, \gamma)$. Put (Gel'fond [4])

\[ \sigma(t) = \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{\varphi_{\gamma}(t)}{\theta^{k-1}}, \quad \tau = \sum_{k=1}^{\infty} \frac{t}{\theta^{k-1}}. \]

Cigler [3] showed that this unique measure $\mu$ is defined by setting

\[ \mu(E) = \int_E \sigma(\alpha) d\alpha \]

for every measurable set $E \subset [0, 1)$.

The explicit formula for the invariant measure has been also found, independently by Parry [17] who additionally remarked that $T$ is even weakly mixing.

Now it is necessary to say a few words about the notation. In the remainder of the § 1 and in § 4 $T$ always means any one of the three transformations and $\mu$ always stands for the invariant measure associated with $T$ as described in Sections 1 ABC. For example Theorem 2 ABC in fact consists of three theorems and should be interpreted to mean that Theorem 2 holds for each of the three transformations and further that in Theorem 2A $\mu(I) = \mu((a, b))$ is an interval—stands for $b - a$, that in Theorem 2B $\mu(I)$ means

\[ \frac{1}{\log 2} \int_a^b \frac{dx}{1 + x} = \frac{1}{\log 2} (\log (1 + b) - \log (1 + a)) \]

and that likewise in Theorem 2C $\mu(I)$ denotes $\int_a^b \sigma(t) dt$. Throughout
the paper "almost all" always means all except a set of Lebesgue-measure 0.

In the first part of this paper some of the common properties of these transformations are investigated. I shall prove:

**Theorem 1** A, B, C. \( T \) is strongly mixing.

Theorem 1A is well known. It holds even for a compact connected abelian group (Hartman, Ryll-Nardzewski [9], p. 169).

This paper was already typed when Professor Krickeberg in a letter kindly called my attention to a paper of Rohlin [20]. Rohlin showed that a wide class of transformations are—what he calls—exact endomorphisms and consequently that they are mixing of every degree which implies strongly mixing. Since the proof I give is different from Rohlin’s proof and since Theorem 1 is a straightforward application of some lemmas used to prove Theorem 2, I did not withdraw Theorem 1. Furthermore its proof can be easily extended to show the mixing property of every degree.

In analogy to some results on Diophantine approximation we get:

**Theorem 2** A, B, C. Let \( T \) be any of the three transformation associated with its invariant measure \( \mu \) as described in A, B, C. Let \((I_n)\) be an arbitrary sequence of intervals contained in the unit interval. For any positive integer \( N \) and \( x \in [0, 1] \) denote by \( A(N, x) \) the number of positive integers \( n \leq N \) such that \( T^n x \in I_n \). Put

\[
\phi(N) = \sum_{n \leq N} \mu(I_n).
\]

Then

\[
A(N, x) = \phi(N) + O(\phi^{1/\varepsilon}(N) \log^{3/\varepsilon+\varepsilon} \phi(N)) \quad \varepsilon > 0
\]

for almost all \( x \in [0, 1] \).

LeVeque [15] has proved theorems of the same type as Theorem 2A for arbitrary sequences \((a_n)\) of integers instead of \((a^n)\) under certain assumptions on the intervals \( I_n \). Recently, his results have been extended by Walker [26]. The novelty in Theorem 2A is the arbitrariness of the intervals \( I_n \); in particular that we can dispense the assumption that the sequence \((\mu(I_n))\) is decreasing. This is not a contradiction to a theorem of Cassels ([1], p. 215) since with Cassels’ notation every subsequence of \((a^n)\) is again a \( \Sigma \)-sequence and so the method of proof of Cassels’ theorem does not apply to our case.

Theorem 2 is a generalization and refinement of some well known results on distribution functions of certain sequences. \( \mu(x) \) is called
the distribution function or distribution measure of the sequence \((x_n)(0 \leq x_n \leq 1)\) if for all \(0 \leq x \leq 1\)

\[
\lim_{n \to \infty} \frac{A(n, x)}{n} = \mu(x).
\]

Here \(A(n, x)\) denotes the number of positive integers \(k \leq n\) such that \(x_k < x\) (see Cigler und Helmberg [4], § 7). In each of the cases ABC the individual ergodic theorem implies at once that the sequence \((T^\alpha)\) has the distribution function \(\mu(x)\) for almost all \(\alpha - \mu(x)\) is the measure of the interval \((0, x) - \mu\) invariant under \(T\). These results are well known (H. Weyl [27], Ryll-Nardzewski [21], Gel’fond [7]). The case A follows also from the fact that \((a^\alpha)\) is uniformly distributed for almost all \(\alpha\). Putting in Theorem 2 \(I_n = I = (0, x)\) for \(n = 1, 2, \cdots\) we get at once

**Corollary ABC.** For \(0 \leq x \leq 1\) denote by \(A(n, \alpha, x)\) the number of positive integers \(k \leq n\) such that \(T^k \alpha < x\). Then for almost all \(\alpha\)

\[
A(n, \alpha, x) = n\mu(x) + O(n^{1/\theta})
\]

where \(\mu(x)\) denotes the \(\mu\)-measure of the interval \((0, x) - \mu\) invariant under \(T\).

2. A quantitative Borel-Cantelli Lemma. Throughout the rest of the paper \(|E|\) denotes the measure of \(E\) in the underlying measure-space.

**Theorem 3.** Let \((E_n, n \geq 1)\) be a sequence of measurable sets in an arbitrary probability space. Denote by \(A(N, x)\) the number of integers \(n \leq N\) such that \(x \in E_n\). Put

\[
\phi(N) = \sum_{n \leq N} |E_n|.
\]

Suppose that there exists a convergent series \(\sum C_k\) with \(C_k \geq 0\) such that for all integers \(n > m\) we have

\[
|E_n \cap E_m| \leq |E_n| |E_m| + |E_n| C_{n-m}.
\]

Then

\[
A(N, x) = \phi(N) + O(\phi^{\frac{1}{2}}(N) \log^{\frac{3}{4} + \epsilon}(N)) \quad \epsilon > 0
\]

for almost all \(x\).

Theorems of this kind have been proved by LeVeque [15] and W.M. Schmidt [22] for particular sequences of sets on the real line.
In an earlier draft I obtained the error term $O(\phi^{2/3}(N) \log^{1/3+\epsilon}(N))$ using a well known device of H. Weyl [27]. However, it was pointed out to me that W.M. Schmidt [22] modification of Rademacher's method for orthogonal sums gives a better estimate. In the following proof I shall use Schmidt's method.

**Proof.** In case that $\phi(\infty) < \infty$ the theorem follows from the convergence part of the Borel-Cantelli lemma even without assuming (2). So we may assume that $\phi(N) \to \infty$. Denote by $\chi(x)$ the characteristic function of $E_n$. For $m < n$ put

$$A(m, n, x) = \sum_{i=m+1}^{n} \chi(x)$$

and

$$\phi(m, n) = \int A(m, n, x) d\mu(x).$$

Then clearly $\phi(0, N) = \phi(N)$ and $A(0, N, x) = A(N, x)$. Using (2) we obtain

\[
\int (A(m, n, x) - \phi(m, n))^2 d\mu(x)
\]

\[
= 2 \sum_{m < i < j \leq n} |E_i \cap E_j| + \phi(m, n) - \phi^2(m, n)
\]

\[
\leq \phi(m, n) - \phi^2(m, n) + 2 \sum_{m < i < j \leq n} |E_i| |E_j| + 2 \sum_{m < i < j \leq n} |E_j| C_{j-i}
\]

\[
\leq \phi(m, n) + 2 \sum_{m < j \leq n} |E_j| \cdot \sum_{m < i < j} C_{j-i}
\]

\[
= O(\phi(m, n)).
\]

(3)

For integer $u \geq 0$ we define $N_u$ to be the largest integer $N$ such that $\phi(N) < u$. Denote by $L_r$ the set of intervals $(u, v]$ with $u = t \cdot 2^s$, $v = (t + 1)2^s \leq 2^r$ where $s \geq 0$, $t \geq 0$ and $r \geq 1$ are integer. Now

$$\sum \phi(N_u, N_s) \leq \phi(N_{2^r}) < 2^r$$

where the summation is extended over all $(u, v] \in L_r$ corresponding to a fixed $s$ since these intervals cover $(0, 2^s]$ exactly once and thus the corresponding intervals $(N_u, N_s]$ cover $(0, N_{2^s}]$ exactly once. Clearly $s \leq r$ and so

$$\sum_{(u,v] \in L_r} \phi(N_u, N_s) < (r + 1)2^r. \quad (4)$$

Put

$$Z_r = Z(r, x) = \sum_{(u,v] \in L_r} (A(N_u, N_s, x) - \phi(N_u, N_s))^2.$$

Then (3) and (4) imply
\[
\int Z_r d\mu(x) = O(r^{2r})
\]
or
\[
\int \frac{Z^r}{2^r r^{2r+1}} d\mu(x) = O(r^{-r}) .
\]
Therefore
\[
(5) \quad Z_r = O(2^r r^{2r})
\]
for almost all \(x\). If \(w\) is an integer and \(2^{-1} < w \leq 2^r \) then \((0, w]\) can be represented as the union of at most \(r\) intervals of \(L_r\) and thus so can \((0, N_w]\). Hence
\[
A(N_w, x) - \phi(N_w) = \sum A(N_u, N_v, x) - \phi(N_u, N_v))
\]
where the sum is over at most \(r + 1\) intervals \((u, v] \in L_r\). This equation together with (5) and Cauchy's inequality yields
\[
(A(N_w, x) - \phi(N_w))^2 = O(2^r r^{3r})
\]
almost everywhere. Hence the theorem is true for all \(N = N_w\). If \(N\) is arbitrary find \(w\) such that \(N_w \leq N < N_{w+1}\). We obviously have
\[
A(N_w, x) \leq A(N, x) \leq A(N_{w+1}, x)
\]
and
\[
\phi(N_w) \leq \phi(N) \leq \phi(N_{w+1}) .
\]
Since
\[
\phi(N_{w+1}) \leq \phi(N_w) + O(1)
\]
the result follows.

3. The overlap estimates.

3A. This section deals with the situation as described in Section 1A.

**Lemma 1.** Let \(P > 1\) be an integer and let \(E = (x_1, x_2)\) be an interval and \(F\) be a measurable set both contained in \([0, 1]\). Then
\[
|E \cap S^{-1}F| = |E| |F| + 2P^{-1} |F| \eta
\]
where \(|\eta| \leq 1\) and \(S\) denotes the transformation \(\alpha \rightarrow \{P\alpha\}\).

**Proof.** If \(\rho(t)\) denotes the characteristic function of \(F\) with period 1 we have
| E \cap S^{-1}F | = \int_{x^2}^{x_1} \rho(Pt)dt = P^{-1} \int_{x^2}^{x_1} \rho(t)dt \\
= |E| \cdot |F| + 2P^{-1} |F| \eta \phantom{1} |\eta| \leq 1.

3B. We use now the notation introduced in Section 1B. We begin with a lemma which is essentially due to Khintchine [10, 11] (see also [12] p. 89).

**Lemma 2.** Let \( E = \{\alpha | a_1(\alpha) = r_1, \ldots, a_k(\alpha) = r_k\} \) and let \( F \) be a measurable set. Then

\[
|E \cap T^{-n-k}F| = |E| \cdot |F| (1 + O(q^{-n})) \phantom{1} q < 1.
\]

**Proof.** For \( 0 < x \leq 1 \) denote by \( \varphi_n(x) \) the \((\mu-)\)measure of the set—Khintchine used the Lebesgue-measure—

\[
E = \{\alpha | a_1(\alpha) = r_1, \ldots, a_k(\alpha) = r_k, T^{n+k}\alpha < x\}.
\]

Then \( \varphi_n(x) \) satisfies the functional equation

\[
\varphi_n(x) = \sum_{g=1}^{\infty} \left( \varphi_{n-1}(\frac{1}{g}) - \varphi_{n-1}(\frac{1}{g + x}) \right).
\]

In fact, \( T^{n+k}\alpha < x \) is equivalent with

\[
0 \leq \frac{1}{T^{n+k-1}\alpha} - \left[ \frac{1}{T^{n+k-1}\alpha} \right] < x
\]

or else with

\[
\frac{1}{g + x} < T^{n+k-1}\alpha \leq \frac{1}{g}
\]

for \( g = 1, 2, \ldots \). But this implies (7).

Now we are going to compute \( \varphi_0(x) \) and its derivatives; \( \varphi_0(x) \) is the measure of the set

\[
M = \{\alpha | a_1(\alpha) = r_1, \ldots, a_k(\alpha) = r_k, T^k\alpha < x\}.
\]

Consequently \( M \) is just the interval with the endpoints \( p_k/q_k, (p_k + p_{k-1}x)/(q_k + q_{k-1}x) \) where \( p_k/q_k = [r_1, \ldots, r_k] \). This follows from the fact that \( \alpha \in M \) is equivalent with

\[
\alpha = [r_1, r_2, \ldots, r_k + T^k\alpha], T^k\alpha < x.
\]

Hence, by computation

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1 Throughout the rest of the paper \( q, p \) and the constant implied by 0 are absolute constants unless otherwise stated.
\[
\log 2 \cdot \varphi_0(x) = (-1)^k \left( \log \left(1 + \frac{p_k + p_{k-1}x}{q_k + q_{k-1}x}\right) - \log \left(1 + \frac{p_k}{q_k}\right)\right)
\]
\[
\log 2 \cdot \varphi'(x) = \left((p_k + q_k)q_k + x((p_{k-1} + q_{k-1})q_k + (p_k + q_k)q_{k-1})\right.
\]
\[+ q_{k-1}(p_{k-1} + q_{k-1})x^2)^{-1}\]
\[
\log 2 \cdot \varphi''(x) = -\frac{(p_{k-1} + q_{k-1})q_k + (p_k + q_k)q_{k-1} + 2q_{k-1}(p_{k-1} + q_{k-1})x}{((p_k + q_k)q_k + x((p_{k-1} + q_{k-1})q_k + (p_k + q_k)q_{k-1} + q_{k-1}(p_{k-1} + q_{k-1})x^2)^2}
\]

and
\[
\log 2 \cdot |E| = (-1)^k \log \left(1 + \frac{(-1)^k}{(p_k + q_k)(q_k + q_{k-1})}\right).
\]

Write
\[
\psi_n(x) = |E|^{-1} \varphi_n(x).
\]
Then \(\psi_n(x)\) also satisfies (7) and by the last set of formulas we obtain
\[
0 < |\psi'(x)| \leq 4 \quad \text{and} \quad |\psi''(x)| < 32 \quad x \in (0, 1].
\]

Hence a theorem of Kuzmin yields (see [10] or [12] p. 78)
\[
\left| \psi_n'(x) - \frac{1}{\log 2} \cdot \frac{1}{1 + x}\right| \leq c_i q^{-\alpha}
\]

where \(c_i\) and \(q < 1\) are absolute positive constants. Integrating from \(a\) to \(b\) we obtain
\[
|\psi_n(b) - \psi_n(a) - |F|| \leq c_i (b - a)q^{-\alpha} \leq c |F| q^{-\alpha}.
\]

This proves the lemma for the case that \(F = (a, b)\) is an interval. Now \(\psi_n(x)\) defines a normed measure \(\psi_n\) on \([0, 1)\) in the usual way. We rewrite the last inequality as
\[
C_n \mu(I) \leq \psi_n(I) \leq C_n \mu(I)
\]

where \(C_n, C\) are constants. It follows that this inequality holds for arbitrary measurable sets \(F \subset [0, 1)\).

Note. If \(\varphi_n(x)\) is defined to be the Lebesgue-measure of the set (6) a theorem of Szüsz [24] gives a sharper estimate. Strangely enough the hypotheses of Szüsz’ theorem are not satisfied in our case.

**Lemma 3.** Let \(F\) be a measurable set and let \(E = (p_k/q_k, p'_k/q'_k)\) where \(p_k/q_k, p'_k/q'_k\) are kth convergents. Then
\[
|E \cap T^{-(n+b)}F| = |E| \cdot |F| (1 + O(q^{-\alpha})) \quad q < 1.
\]
Proof. This is an immediate consequence of Lemma 2 since $E$ is the union of at most countably many disjoint intervals for which the partial quotients $a_j(\alpha)$, $j \leq k$ are constant.

**Lemma 4.** Let $E$ be an interval and $F$ be a measurable set. Then

$$|E \cap T^{-n}F| = |E| |F| + |F| 0(q^{-n}) \quad q < 1.$$ 

**Proof.** Put $k = \lceil n/2 \rceil$ and let $E = (x, y)$. Then there exist convergents $p_k/q_k, p'_k/q'_k$ such that $x$ and $y$ are contained in intervals with endpoints

$$x \in \left(\frac{p_k}{q_k}, \frac{p_k + p_{k-1}}{q_k + q_{k-1}}\right), \quad y \in \left(\frac{p'_k}{q'_k}, \frac{p'_k + p'_{k-1}}{q'_k + q'_{k-1}}\right)$$

respectively. Call the intersection of $E$ with these two intervals $E_1$ and $E_2$. Then $E = E_0 \cup E_1 \cup E_2$ where $E_0$ is of type described in Lemma 3. Hence by Lemma 3 and using the $T$ preserves $\mu$

$$|E \cap T^{-n}F| - |E| |F| \leq C |E_0| |F| q^{-k} + |F| (|E_1| + |E_2|).$$

But for $i = 1, 2$

$$|E_i| \leq \frac{1}{q_k} \leq \frac{3}{2} \left(\frac{2}{3}\right)^i$$

Observing that $q < 1$ implies $q^{n/2} < 1$ we get the result.

Later we need

**Lemma 5.** Let $E = \{\alpha \mid a_i(\alpha) \geq M\}$ and $F = \{\alpha \mid a_i(\alpha) \geq N\}$ for positive integers $M, N$. Then

$$|E \cap T^{-n}F| = |E| |F| (1 + O(q^{-n})) \quad q < 1.$$ 

**Proof.** This follows at once from Lemma 2 since for example

$$E = \bigcup_{i = -M}^{\infty} \{\alpha \mid a_i(\alpha) = i\}.$$ 

3C. In this section we use the notation introduced in section 1C. In the next lines $\lambda(E)$ stands for the Lebesgue-measure of $E$.

**Lemma 6.** (Gel'fond [7]) For $t \leq 1$ let $E = [0, t)$ be an interval and let $F$ be a measurable set. Then

$$\lambda(E \cap T^{-n}F) = \lambda(E) |F| + |F| O(\rho^{-n}) \quad 1 < \rho < \theta.$$ 

**Proof.** Apart from the factor $|F|$ before the 0 symbol this is
just formula (12) in Gel’fond’s paper. But inspection of its proof shows that we may pull out the factor \( |F| \) of the 0-symbol and this symbol still has the required properties.

**Lemma 7.** Let \( E \) be an interval and let \( F \) be a measurable set. Then
\[
|E \cap T^{-n}F| = |E||F| + |F| O(\rho^{-n}) \quad \rho > 1.
\]

**Proof.** It is enough to show the lemma in case that \( E = [0, t) \). Let \( f \) be the characteristic function of \( F \). Using Lemma 6 we obtain
\[
|E \cap T^{-n}F| = \int_0^t \varphi_n(\alpha)f(x_n(\alpha))\sigma(\alpha)d\alpha
\]
\[
= \frac{1}{\tau} \sum_{k=1}^\infty \frac{1}{\theta^{k-1}} \int_0^t \varphi_n(\alpha)f(x_n(\alpha))\varphi_{t_k}(\alpha)d\alpha
\]
\[
= \frac{1}{\tau} \sum_{k=1}^\infty \frac{1}{\theta^{k-1}} \int_0^{\min(t, t_k)} f(x_n(\alpha))d\alpha
\]
\[
= \frac{1}{\tau} |F| \sum_{k=1}^\infty \min(t, t_k) + |F| O(\rho^{-n})
\]
\[
= |E||F| + |F| O(\rho^{-n}).
\]

4. Proof of Theorem 1 and 2.

4.1 **Proof of Theorem 1.** We have to show that for every pair of measurable sets \( A_1, A_2 \subset [0, 1] \)
\[
\lim_{n \to \infty} |A_1 \cap T^{-n}A_2| = |A_1||A_2|.
\]

We approximate \( A_i \) \((i = 1, 2)\) in measure by a finite union \( E_i \) \((i = 1, 2)\) of disjoint intervals arbitrarily closely: Given \( \varepsilon > 0 \) we can find \( E_i \) such that
\[
|A_i \Delta E_i| < \varepsilon \quad (i = 1, 2).
\]

By Lemmas 1, 4, 7 we easily get for all sufficiently large \( n \)
\[
||E_i \cap T^{-n}E_z|-|E_i||E_z|| < \varepsilon.
\]

Using that \( T \) preserves \( \mu \) we have
\[
||A_1 \cap T^{-n}A_2|-|A_1||A_2|\| \leq ||A_1 \cap T^{-n}A_2|-|E_i \cap T^{-n}A_z||
\]
\[
+ ||E_i \cap T^{-n}A_z|-|E_i \cap T^{-n}E_z||
\]
\[
+ ||E_i \cap T^{-n}E_z|-|E_i||E_z|| + ||E_i||E_z|| - |A_1||A_2| < 5\varepsilon
\]
for all sufficiently large \( n \).

Rohlin’s theorem that \( T \) is mixing of every degree follows in the same way from:
LEMMA 8ABC. Let \( n_1, \ldots, n_r \) be positive integers and let \( I_1, I_2, \ldots, I_r \) be intervals all contained in the unit interval. Then
\[
|I_0 \cap T^{-n_1}I_1 \cap \cdots \cap T^{-n_r}I_r|
= \prod_{i=0}^r |I_i| + |I_r| \cdot O(q^{\min \{n_i\}}) \quad q < 1
\]
where the constant implied by \( O \) only depends on \( r \).

For a proof we only have to apply Lemmas 1, 4, 7 several times.

4.2. In order to prove Theorem 2 we put \( E_n = T^{-n}I_n \). Using Lemmas 1, 4, 7 and the fact that \( T \) preserves \( \mu \) we obtain for \( n > m \)
\[
|E_n \cap E_m| = |I_m \cap T^{-(n-m)}I_n|
\leq |I_m| |I_n| + |I_n| \cdot O(q^{-n-m})
= |E_m| |E_n| + |E_n| \cdot O(q^{-n-m}) \quad q < 1.
\]
Observing that all the measures involved are equivalent to the Lebesgue-measure we get Theorem 2 as an application of Theorem 3.

5. Some metrical theorems on continued fractions. In this section some metrical theorems on continued fractions are proved. I use the same notation as in sections 1B and 3B. The main result (Theorem 4) is a refinement of a theorem of Khintchine [11]. LeVeque [14] has outlined a proof of a weaker form of Theorem 4. Several applications of Theorem 4 are given. Finally a well known theorem (e.g. see [12], p. 67) of Bernstein is sharpened.

Again it will be illustrated that it is more natural to use the measure \( \mu \) invariant under \( T \) rather than the Lebesgue-measure. In the theorems though "almost all" always means "all except a set of Lebesgue-measure 0." But \( \mu \) and the Lebesgue-measure have the same null-sets and hence-considered from this point of view it makes no difference which of them we use.

5.1. The general theorem.

THEOREM 4. Let \( f(u_1, \ldots, u_k) \) be a nonnegative function defined for all \( k \)-tuples \((u_1, \ldots, u_k)\) with positive entries and satisfying
\[
(8) \quad \int_0^1 (f(a_1(x), \ldots, a_k(x)))^2 dx < \infty.
\]
Then for integers \( n \geq 1 \) and fixed \( k \geq 1 \)
\[
\frac{1}{n} \sum_{i=0}^{n-1} f(a_i(T^i x), \ldots, a_k(T^i x)) \\
= \frac{1}{\log 2} \int_0^1 f(a_i(x), \ldots, a_k(x)) \frac{dx}{1 + x} + O(n^{-1/2} \log^{3/2 + \epsilon} n)
\]

almost everywhere.

If \( k \) is not fixed but if instead \( n \) and \( k \) are linked by

\[
2^k \leq n < 2^{k+1}
\]

then (9) holds with \( n^{-1/2} \log^{2 + \epsilon} n \) instead of \( n^{-1/2} \log^{3/2 + \epsilon} n \) in the error term.

**Proof.** Since \( T \) preserves \( \mu \) we have for integer \( i \geq 0 \)

\[
\int_0^1 f(a_i(T^i x), \ldots, a_k(T^i x))d\mu(x) = \alpha_i = \alpha < \infty
\]

by (8). Further for integers \( 0 \leq i < j \) and \( m \geq 0 \)

\[
I(i, j) = \int_0^1 f(a_i(T^{i+m} x), \ldots, a_k(T^{i+m} x)) \cdot f(a_i(T^{j+m} x), \ldots, a_k(T^{j+m} x))
\]

\[
a_k(T^{j+m} x) d\mu(x) = \sum_{r} f(r, \ldots, r_k) \cdot f(r'_i, \ldots, r'_k)
\]

\[
\cdot |E \cap T^{-(j-i)} F|
\]

where the summation is extended over all the \( r \)'s and

\[
E = \{ \alpha; a_i(\alpha) = r_i, \ldots, a_k(\alpha) = r_k \}
\]

and

\[
F = \{ \alpha; a_i(\alpha) = r'_i, \ldots, a_k(\alpha) = r'_k \}.
\]

By Lemma 2 we obtain for \( j > i + k \)

\[
I(i, j) = \sum_{r} f(r, \ldots, r_k) |E| \cdot \sum_{r'} f(r'_i, \ldots, r'_k) |F'| \cdot (1 + O(q^{j-i-k}))
\]

\[
= \alpha^2 (1 + O(q^{j-i-k})).
\]

By (8) we clearly have for \( j \leq i + k \)

\[
I(i, j) \leq \alpha^2.
\]

Hence we obtain by (11)
\[ \int \left( \sum_{i=0}^{n-1} f(a_i(T^{i+m}x), \ldots, a_k(T^{i+m}x)) - \alpha n \right)^2 d\mu(x) \]

(12) \[ = \sum_{i \neq j=0}^{n-1} \int f(a_i(T^{i+m}x), \ldots, a_k(T^{i+m}x)) \cdot f(a_i(T^{j+m}x), \ldots, a_k(T^{j+m}x)) d\mu(x) - \alpha^2 n^2 \leq \alpha^2 \cdot O\left( \sum_{i,j=0}^{n-1} q^{j-i} \right) + \alpha^2 kn = O(n) \]

if \( k \) is fixed. If \( n \) and \( k \) are linked by (10) then we get in (12) only the estimate \( O(n \log n) \). In both cases Theorem 6 (p. 649) of Gál and Koksma [6] gives the result.

De Vroedt [25] has obtained earlier a result of this type also for a narrow class of functions only. His theorem and mine have a non-empty overlap: To see this put in Theorem 4 \( k = 1 \) and \( f(u_i) = f(u) - \log u \). We get

**COROLLARY 1 (de Vroedt).** Almost everywhere

\[ (a_1(x) \cdots a_n(x))^{1/n} = \prod_{i=1}^{m} \left( 1 + \frac{1}{n^2 + 2n} \right)^{\log n / \log 2} + O(n^{-1/2} \log^{3/2 + \varepsilon} n) \]

This is a refinement of a well-known theorem of Khintchine [10]. Applying Theorem 1 for the characteristic function of the set \( \{ x : a_1(x) = p \} \) we get a refinement of a theorem of Lévy [19] on the frequency of the digit \( p \).

**COROLLARY 2 (de Vroedt).** Denote by \( h(n, p, x) \) the number of positive integers \( k \leq n \) such that \( a_k(x) = p \). Then for almost all \( x \)

\[ h(n, p, x) = \frac{1}{\log 2} \log \left( 1 + \frac{1}{p(p + 2)} \right) + O(n^{-1/2} \log^{3/2 + \varepsilon} n) \]

Doeblin [5] using quite different methods sketched a proof of the law of the iterated logarithm in both cases. Recently, Stackelberg [23] has announced a proof different from that of Doeblin.

As another application of Theorem 4 I shall prove:

**THEOREM 5.** Denote by \( q_n(x) \) the denominator of the \( n \)th convergent to \( x \). Then

\[ \sqrt[n]{q_n(x)} = \exp (\pi^2/12 \log 2) + O(n^{-1/2} \log^{3/2 + \varepsilon} n) \]

almost everywhere.

This improves slightly LeVeque's [14] refinement of a well-known theorem of Khintchine [11] and Lévy [17]. Furthermore the proof will not depend on Lévy's result. For this reason we need some lemmas.
5.2. Some lemmas. The first one can be proved by induction or by taking the transposed matrices.

**Lemma 9.** If

\[
\begin{pmatrix}
a_k & 1 \\
1 & 0 \\
\end{pmatrix} \cdots \begin{pmatrix}
a_1 & 1 \\
1 & 0 \\
\end{pmatrix} = \begin{pmatrix}
p_k & q_k \\
p_{k-1} & q_{k-1} \\
\end{pmatrix}
\]

then

\[
\begin{pmatrix}
a_1 & 1 \\
1 & 0 \\
\end{pmatrix} \cdots \begin{pmatrix}
a_k & 1 \\
1 & 0 \\
\end{pmatrix} = \begin{pmatrix}
q_{k-1} & q_k \\
p_{k-1} & p_k \\
\end{pmatrix}.
\]

If we consider \(a_1, \ldots, a_k\) as partial quotients of a continued fraction then \(p_k/q_k\) is just the \(k\)th convergent. So the lemma describes the relation between the two fractions \([a_1, \ldots, a_k]\) and \([a_k, \ldots, a_1]\). This is used in

**Lemma 10.** Let

\[E = \{x \mid a_i(x) = r_i, \ldots, a_k(x) = r_k\}\]

and

\[F = \{x \mid a_i(x) = r_k, \ldots, a_k(x) = r_1\}\]

then

\[|E| = |F| .\]

**Proof.** Put \([r_1, \ldots, r_k] = p_k/q_k\) and \([r_k, \ldots, r_1] = p'_k/q'_k\) then

\[
\log 2 \cdot |E| = (-1)^k \log \left(1 + \frac{(-1)^k}{(p_k + q_k)(q_k + q_{k-1})}\right)
\]

and similarly for \(|F|\) with primes. But Lemma 9 yields \(q_k = q'_k\), \(p_k = q_{k-1}\), and \(q'_k = p_k\). Hence the result.

**Lemma 11.** Define for \(k \geq 1\)

\[
f_k(x) = a_k(x) + [a_{k-1}(x), \ldots, a_1(x)] \text{ if}\]

\[x = [a_1(x), \ldots, a_k(x), \ldots] \text{-irrational}
\]

and

\[= 0 \text{ if } x \text{ is rational}\]

then

\[
\lim_{k \to \infty} \int_0^1 \log f_k(x) d\mu(x) = \frac{\pi^2}{12 \log 2} .
\]
Proof. Using Lemma 10 we obtain

$$\lambda_k = \int_0^1 \log f_k(x) \, d\mu(x)$$

$$= \sum_{r_1, \ldots, r_k \geq 1} \log (r_k + [r_{k-1}, \ldots, r_1]) \mu(\{\alpha \mid a_i(\alpha) = r_i, \ldots, a_k(\alpha) = r_k\})$$

$$= \sum_{r_1, \ldots, r_k \geq 1} \log (r_1 + [r_2, \ldots, r_k]) \mu(\{\alpha \mid a_i(\alpha) = r_i, \ldots, a_k(\alpha) = r_k\})$$

$$= \int_0^1 \log (a_k(x) + [a_{k-1}(x), \ldots, a_1(x)]) \, d\mu(x).$$

But $\log (a_k(x) + [a_{k-1}(x), \ldots, a_1(x)]) \leq \log (a_k(x) + 1)$ which is integrable. Hence

$$\lim_{k \to \infty} \int_0^1 \log f_k(x) \, d\mu(x) = \frac{1}{\log 2} \int_0^1 \log \frac{1}{x} \cdot \frac{dx}{1 + x} = \frac{\pi^2}{12 \log 2}.$$  

5.3. Now we can finish the proof of Theorem 5. Essentially, we use the argument of Khintchine [11]. Let $k \geq 1$ be an integer. Define for $n \geq k$

$$f_n^{(k)}(x) = a_n(x) + [a_{n-1}(x), \ldots, a_{n-k+1}(x)] \text{ if } x = [a_1(x), \ldots, a_n(x), \ldots] \text{ irrational}$$

$$= 0 \text{ if } x \text{ is rational.}$$

Then $f_n^{(k)}(Tx) = f_{n+1}^{(k)}(x)$. Hence for $n \geq k$

$$f_n^{(k)}(x) = f_k^{(k)}(T^{n-k}x) = f_k(T^{n-k}x)$$

in the notation of Lemma 11.

We observe

$$\frac{q_n(x)}{q_{n-1}(x)} = f_n(x).$$

For $n \geq k$

$$|\log f_n(x) - \log f_n^{(k)}(x)| \leq |f_n(x) - f_n^{(k)}(x)| \leq 2^{-k}$$

by Lemma 4, p. 7 in Cassel’s book [2]. Using again this lemma and the first part of the proof of Lemma 11 we get

$$\left| \lambda_k - \frac{\pi^2}{12 \log 2} \right| \leq \int_0^1 |\log (a_1(x) + [a_2(x), \ldots, a_k(x)]) - \log (a_1(x) + [a_2(x), \ldots])| \, d\mu(x)$$

$$+ \int_0^1 |[a_3(x), \ldots, a_k(x)] - [a_3(x), \ldots]| \, d\mu(x)$$

$$\leq 2^{-k}.$$
From now on \( n \) and \( k \) are linked by (10). Since 
\[
\int_0^1 \left( \frac{1}{n^{1/2}} \sum_{i=1}^{k-1} \log f_i(x) \right)^2 \, dx = O(n \log^2 n)
\]
we obtain by the mentioned theorem of Gáll and Koksma

\[
(16) \quad \frac{1}{n} \sum_{i=1}^{k-1} \log f_i(x) = O(n^{-1} \log^{5/2+\varepsilon})
\]
almost everywhere. Hence by Theorem 4, (10), (14), (15), (16)
\[
\left| \frac{1}{n+k-1} \sum_{i=1}^{n+k-1} \log f_i(x) - \frac{\pi^2}{12 \log 2} \right| < \frac{k-1}{n} \frac{\pi^2}{12 \log 2} + \left| \frac{1}{n} \sum_{i=1}^{k-1} \log f_i(x) \right| + \frac{1}{n} \sum_{i=k}^{n+k-1} \left| \log f_i(x) \right|
\]
\[
- \log f_i(x) \right| + \left| \frac{1}{n} \sum_{i=k}^{n+k-1} \log f_i(x) - \lambda_k \right| + \left| \lambda_k - \frac{\pi^2}{12 \log 2} \right|
\]
\[
= O(n^{-1/2} \log^{5/2+\varepsilon})
\]
almost everywhere. Since obviously every positive integer \( N > 2 \) can
be written as \( N = n + k \) (\( n, k \) subject to (10)) (13) yields the result.

It might be interesting to remark that Khintchine-Lévy’s theorem,
namely that
\[
V_n(x) = \exp \left( \frac{\pi^2}{12 \log 2} \right) + O(n^{-1/2} \log^{5/2+\varepsilon})
\]
almost everywhere, is another interesting application of the individual ergodic theorem. The proof
is apart from a few simplifications the same as that of Theorem 2.

**Corollary.** If \( p_n(x) \) denotes the numerator of the \( n \)th convergent to \( x \) then almost everywhere

\[
V_n(x) = \sqrt[n]{x} \cdot \exp \left( \frac{\pi^2}{12 \log 2} \right) + O(n^{-1/2} \log^{5/2+\varepsilon})
\]

This follows from the inequality

\[
\left| \sqrt[n]{p_n(x)} - \sqrt[n]{\sqrt[n]{q_n(x)}} \right| \leq \frac{2}{n}
\]
for \( 0 < x \leq 1 \).

5.4. I shall prove now a refinement of a well known theorem of
Bernstein (e.g. see [12] p. 67). However, Doeblin [5] states with
some misprints that even the law of the iterated logarithm holds in
case that \( \varphi(n) \to \infty \).

**Theorem 6.** Let \( (\varphi(n)) \) be any sequence of positive integers and
denote by \( A(N, x) \) the number of positive integers \( n \leq N \) such that
\( a_n(x) \geq \varphi(n) \). Put
\[ \phi(N) = \frac{1}{\log 2} \sum_{n \leq N} \log \left( 1 + \frac{1}{\varphi(n)} \right). \]

Then for almost all \( x \)
\[ A(N, x) = \phi(N) + O(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N)). \]

**Proof.** Put
\[ E_n = \{ \alpha \mid a_n(\alpha) \geq \varphi(n) \} \quad \text{and} \quad U_n = T^{-(n-1)}E_n. \]

Then
\[ U_n = \{ \alpha \mid a_n(\alpha) \geq \varphi(n) \}. \]

Using Lemma 5 and the fact that \( T \) preserves \( \mu \) we get for \( n > m \)
\[
\begin{align*}
|E_n \cap E_m| &= |U_m \cap T^{-(n-m)}U_n| \leq |U_m| |U_n| (1 + O(q^{\varphi(n-m)})) \\
&= |E_m| |E_n| (1 + O(q^{\varphi(n-m)})).
\end{align*}
\]

But
\[ |E_n| = |U_n| = \frac{1}{\log 2} \int_0^{1/\varphi(n)} \frac{dx}{1 + x} = \frac{1}{\log 2} \log \left( 1 + \frac{1}{\varphi(n)} \right). \]

Since \( \mu \) is equivalent to the Lebesgue-measure Theorem 6 follows from Theorem 3.

**References**

12. ———, *Kettenbrüche*, Teubner, Leipzig, 1956 (there is also an English translation).

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