A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED PROOF OF LEBESGUE’S THEOREM

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A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED PROOF OF LEBESGUES THEOREM

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The main theorem in this paper is on (not necessarily commutative) lattice ordered groups, and is a generalization of a result on finitely additive set functions due to Namioka. Our result can be used to prove Ptak's combinatorial theorem on convex means, to give a short non measure-theoretic proof of Lebesgue's dominated convergence theorem for a sequence of continuous functions on a countably compact topological space, and to give a short proof of Banach's criteria for the weak convergence of a sequence in the Banach space of all bounded, real functions on an abstract set.

We shall prove the following result.

**Theorem 1.** If \( L \) is a lattice ordered group, \( \{g_1, g_2, \ldots \} \) is a sequence of positive elements in \( L \), and \( \varphi \) is an order-preserving homomorphism of \( L \) into the real numbers such that the sequence \( \varphi(g_1 \vee \cdots \vee g_p)(p = 1, 2, \cdots) \) is bounded above and \( \lim \sup_p \varphi(g_p) > 0 \) then there exist integers \( 0 < r(1) < r(2) < \cdots \) such that, for each \( s \), \( \varphi(g_{r(1)} \wedge \cdots \wedge g_{r(s)}) > 0 \).

The idea for this stems from the following result of Namioka.

**Namioka's Theorem.** If \( X \) is a nonvoid set, \( S \) is a field of subsets of \( X \), \( \{A_1, A_2, \ldots \} \) is a sequence in \( S \) and \( \mu \) is a positive, finitely additive function on \( S \) such that \( \lim \sup \mu(A_p) > 0 \) then there exist integers \( 0 < r(1) < r(2) < \cdots \) such that, for each \( s \), \( \mu(A_{r(1)} \cap \cdots \cap A_{r(s)}) > 0 \).

Namioka's Theorem can be found in [2, 17.9, p. 157] and [4, Lemma 2, p. 714]—it can clearly be deduced from our result by taking \( L \) to be the set of all \( S \)-simple functions and \( \varphi(\cdot) = \int \cdot \ d\mu \).

Namioka's theorem was proved in order to give a proof of Krein's Theorem on weak compactness that avoids measure theory. This has also been done, in a superficially very different way, by Ptak, using his combinatorial theorem on the existence of convex means, which appears in print in [3, §24, No. 6, p. 331], [5, 1.3, p. 439] and [6]. Ptak shows that Namioka's theorem can be deduced from his ([5, 5.9, p. 447]—Ptak proves a slightly weaker form in which the conclusion "\( \mu(A_{r(1)} \cap \cdots \cap A_{r(s)}) > 0 \)" is replaced by "\( A_{r(1)} \cap \cdots \cap A_{r(s)} \neq \emptyset \)").
Ptak’s Theorem. We suppose that $K$ is an infinite set and that $X$ is a nonvoid family of subsets of $K$. We write $P(K)$ for the collection of all positive, real valued functions $\lambda$ on $K$ such that $\{k: k \in K, \lambda(k) > 0\}$ is finite and $\sum_{k \in K} \lambda(k) = 1$; for $x \subseteq K$ we write $\lambda(x) = \sum_{k \in x} \lambda(k)$. If
\[
\inf_{\lambda \in P(K)} \sup_{x \in K} \lambda(x) > 0
\]
then there exist $x_1, x_2, \ldots \in X$ and distinct $k_1, k_2, \ldots \in K$ such that, for each $s$, $\{k_1, \ldots, k_s\} \subseteq x_s$.

We shall show that Ptak’s Theorem can be deduced from Theorem 1 in a natural way and that the “convexity” is a consequence of the result that weak and the norm closures of a convex subset of a normed linear space coincide.

If $X$ is a nonvoid set we write $B(X)$ for the set of all bounded, real functions on $X$. $B(X)$ is a Banach space under the norm $\|f\| = \sup_{x \in X} |f(x)|$. We shall show that Theorem 1 can be used to give criteria for the weak convergence of a sequence in $B(X)$.

Finally, we shall show how Theorem 1 can be used to give a short non measure-theoretic proof of Lebesgue’s dominated convergence theorem for a sequence of continuous functions on a countably compact topological space.

Theorem 1 was first proved in a different context. I would like to thank Professor K. Fan for reading the manuscript and suggesting the possibility of an application to Lebesgue’s theorem.

2. Proof of Theorem 1. Using the identities $f - f \lor g + g \leq f \land g$ and $(f \lor g) \land h = (f \land h) \lor (g \land h) \leq f \land h + g \land h$, valid for any positive $f, g,$ and $h$ in $L$, we see that, if $q, p$ are integers and $0 < q < p$,
\[
\varphi(g_p) + \varphi(g_1 \lor \cdots \lor g_q) \leq \varphi((g_1 \lor \cdots \lor g_q) \lor g_p) + \varphi((g_1 \lor \cdots \lor g_q) \land g_p) \\
\leq \varphi(g_1 \lor \cdots \lor g_p) + \sum_{1 \leq r \leq q} \varphi(g_r \land g_p).
\]

Letting $p \to \infty$,
\[
\lim \sup_p \varphi(g_p) + \varphi(g_1 \lor \cdots \lor g_q) \leq \lim_p \varphi(g_1 \lor \cdots \lor g_p) + \sum_{1 \leq r \leq q} \lim \sup_p \varphi(g_r \land g_p).
\]

Letting $q \to \infty$,
\[
\lim \sup_p \varphi(g_p) + \lim_q \varphi(g_1 \lor \cdots \lor g_q) \leq \lim_p \varphi(g_1 \lor \cdots \lor g_p) + \sum_{r \geq 1} \lim \sup_p \varphi(g_r \land g_p).
\]

By hypothesis, $\lim_p \varphi(g_1 \lor \cdots \lor g_p) = \lim_p \varphi(g_1 \lor \cdots \lor g_p) < \infty$ and
lim sup_p φ(g_p) > 0. Hence

\[ 0 < \sum_{r \geq 1} \lim \sup \phi(g_r \land g_p) \]

and thus there exists r(1) such that lim sup_p φ(g_{r(1)} \land g_p) > 0 — and from this, φ(g_{r(l)}) > 0.

Performing a similar argument on the elements g_{r(1)} \land g_{r(1)+1}, \ldots we can show that there exists r(2) > r(1) such that lim sup_p φ((g_{r(1)} \land g_{r(2)}) \land (g_{r(1)} \land g_p)) > 0, i.e.,

\[ \lim \sup_p \phi(g_{r(1)} \land g_{r(2)} \land g_p) > 0 \]

Continuing this process inductively, we complete the proof of the theorem.

3. Application of Theorem 1 to Ptak’s theorem. If ψ is a bounded linear functional on B(X) then there exists a positive linear functional φ on B(X) such that, for all f ∈ B(X), |ψ(f)| ≤ φ(|f|). (If f ≥ 0 we define φ(f) to be sup_{|g| ≤ f} ψ(g) and extend φ to the whole of B(X) by linearity).

**COROLLARY 2.** Let X be a nonvoid set and Y a bounded subset of B(X). Let Z be the convex extension of Y in B(X). Then (a) implies (b) and (b) implies (c).

(a) Inf_{y ∈ Z} ||z|| > 0.

(b) There exists a positive linear functional φ on B(X) such that inf_{y ∈ Z} φ(|y|) > 0.

(c) For each sequence \{y_1, y_2, \ldots\} in Y there exist integers 0 < r(1) < r(2) < \ldots and x_1, x_2, \ldots ∈ X such that y_{r(1)}(x_s) ≠ 0 whenever 0 < t ≤ s.

**Proof of Corollary 2.** If (a) is true then 0 ∈ norm-closure of Z. Thus, since Z is convex, 0 ∈ weak-closure of Z, hence 0 ∈ weak-closure of Y. Thus there exists a bounded linear functional ψ on B(X) such that inf_{y ∈ X} |ψ(y)| > 0. If the positive linear functional φ on B(X) is chosen as in the remarks preceding this Corollary then (b) is satisfied.

If (b) is true, φ is as in (b) and y_1, y_2, \ldots ∈ Y then lim sup_p φ(|y_p|) > 0. We apply Theorem 1 to L = B(X) and g_p = |y_p| and find that there exist integers 0 < r(1) < r(2) < \ldots such that, for each s, φ(|y_{r(s)}| \land \ldots \land |y_{r(s)}|) > 0. It follows from this that (c) is satisfied.

**Remark.** It can easily be seen that, if X, Y and Z are as above and φ is a positive linear functional on B(X) then
and so, if all the functions in $Y$ are positive, condition (b) of the Corollary implies condition (a). This is, essentially, the proof used in [5, 5.9, p. 447].

Proof of Ptak's Theorem. If $K$ is any nonvoid set and $X$ is any nonvoid family of subsets of $K$ then, for each $k \in K$, we define $y_k \in B(X)$ by $y_k(x) = 0$ if $k \notin x$ and $y_k(x) = 1$ if $k \in x$. We apply Corollary 2 ((a) implies (c)) to $Y = \{y_k; k \in K\}$ and obtain: if

$$\inf_{\lambda \in P(K)} \sup_{x \in X} \lambda(x) > 0$$

then, for all $k_1, k_2, \ldots \in K$, there exist integers $0 < r(1) < r(2) < \cdots$ and $x_1, x_2, \ldots \in X$ such that, for each $s$, $\{k_{r(i)}, \ldots, k_{r(s)}\} \subset x_s$. This is formally stronger than, though in fact equivalent to, Ptak's Theorem.

4. Application of Theorem 1 to Lebesgue's theorem of dominated convergence.

Definition 3. Let $X$ be a nonvoid set and $f_1, f_2, \ldots \in B(X)$. We shall say that $\{f_1, f_2, \ldots\}$ is a Dini sequence if $|f_1| \wedge \cdots \wedge |f_s| \to 0$ uniformly on $X$ as $s \to \infty$.

Corollary 4. Let $X$ be a nonvoid set, $\{f_1, f_2, \ldots\}$ be a sequence in $B(X)$ with the property that all its subsequences are Dini sequences and $\varphi$ be a positive linear functional on $B(X)$ such that the sequence $\varphi(|f_1| \vee \cdots \vee |f_s|)$ $(s = 1, 2, \ldots)$ is bounded above. Then $\varphi(f_n) \to 0$ as $n \to \infty$.

Proof. Let $\varepsilon > 0$ be given. For each $p$ we write $g_p = (|f_p| - \varepsilon 1) \vee 0$. If $0 < r(1) < r(2) < \cdots$ are integers then, by hypothesis, for all sufficiently large $s$, $g_{r(1)} \wedge \cdots \wedge g_{r(s)} = 0$ hence

$$\varphi(g_{r(1)} \wedge \cdots \wedge g_{r(s)}) = 0.$$ 

Thus, from Theorem 1, $\lim \sup_p \varphi(g_p) = 0$.

Since $|f_p| - \varepsilon 1 \leq g_p$, it follows that $\lim \sup_p \varphi(|f_p| - \varepsilon 1) \leq 0$ and hence that $\lim \sup_p \varphi(|f_p|) \leq \varepsilon \varphi(1)$. Since $\varepsilon$ is arbitrary, this implies that $\lim \sup_p \varphi(|f_p|) = 0$. The required result follows since $|\varphi(f_n)| \leq \varphi(|f_n|)$.

If $X$ is a countably compact topological space we write $C(X)$ for the set of all real, continuous functions on $X$. Corollary 4 is, in fact, true with "$B(X)$" replaced everywhere by "$C(X)$"—the proof is identical. In fact any positive linear functional on $C(X)$ can be extended by the Hahn-Banach Theorem to one on $B(X)$ and so the result for $C(X)$ can also be deduced from the result for $B(X)$. If $f_1, f_2, \ldots$
$e \in C(X)$ and $f_n \to 0$ pointwise as $n \to \infty$ then, from Dini’s Theorem, every subsequence of $\{f_1, f_2, \cdots\}$ is a Dini sequence. The following result is then immediate from the “$C(X)$” version of Corollary 4.

**Lebesgue’s Theorem.** If $X$ is a countably compact topological space, $f_1, f_2, \cdots \in C(X)$, $f_n \to 0$ pointwise as $n \to \infty$, and $\varphi$ is a positive linear functional on $C(X)$ such that the sequence $\varphi(|f_1| \vee \cdots \vee |f_p|) (p = 1, 2, \cdots)$ is bounded above, then $\varphi(f_n) \to 0$ as $n \to \infty$.

Our final result is a slightly expanded form of a theorem due to Banach. (See [1, Annexe, §2, Theorem 5, p. 219] and [5, 5.4, p. 445].)

**Banach’s Theorem.** Let $X$ be a nonvoid set and $f_1, f_2, \cdots$ be a bounded sequence in $B(X)$. Then the conditions (a)–(d) are equivalent.

(a) If $x_1, x_2, \cdots \in X$ then $\lim_n \liminf_i |f_n(x_i)| = 0$.
(b) Every subsequence of $\{f_1, f_2, \cdots\}$ is a Dini sequence.
(c) $\varphi(f_n) \to 0$ for every positive linear functional $\varphi$ on $B(X)$.
(d) $f_n \rightharpoonup 0$ weakly in $B(X)$.

**Proof.** If follows from the definition of a Dini sequence that (a) implies (b), from Corollary 4 that (b) implies (c), and from the remarks preceding Corollary 2 that (c) implies (d).

If (a) is false and $x_1, x_2, \cdots \in X$ are such that

$$\lim \sup_n \liminf_i |f_n(x_i)| \geq \varepsilon > 0$$

then there exist integers $0 < n_1 < n_2 < \cdots$ such that, for each $k$, $\liminf_i |f_{n_k}(x_i)| > \varepsilon$. By the diagonal process we can find integers $0 < i(1) < i(2) < \cdots$ such that, for each $k$, $\lim_j (f_{n_k}(x_{i(k,j)}))$ exists and has absolute value greater than $\vareferences


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<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leonard Daniel Baumert, <em>Extreme copositive quadratic forms. II</em></td>
<td>1</td>
</tr>
<tr>
<td>Edward Lee Bethel, *A note on continuous collections of disjoint</td>
<td>21</td>
</tr>
<tr>
<td>continua*</td>
<td></td>
</tr>
<tr>
<td>Delmar L. Boyer and Adolf G. Mader, *A representation theorem for</td>
<td>31</td>
</tr>
<tr>
<td>abelian groups with no elements of infinite p-height*</td>
<td></td>
</tr>
<tr>
<td>Jean-Claude B. Derderian, <em>Residuated mappings</em></td>
<td>35</td>
</tr>
<tr>
<td>Burton I. Fein, <em>Representations of direct products of finite groups</em></td>
<td>45</td>
</tr>
<tr>
<td>John Brady Garnett, <em>A topological characterization of Gleason parts</em></td>
<td>59</td>
</tr>
<tr>
<td>Herbert Meyer Kamowitz, *On operators whose spectrum lies on a</td>
<td>65</td>
</tr>
<tr>
<td>circle or a line*</td>
<td></td>
</tr>
<tr>
<td>Ignacy I. Kotlarski, *On characterizing the gamma and the normal</td>
<td>69</td>
</tr>
<tr>
<td>distribution*</td>
<td></td>
</tr>
<tr>
<td>Yu-Lee Lee, <em>Topologies with the same class of homeomorphisms</em></td>
<td>77</td>
</tr>
<tr>
<td>Moshe Mangad, *Asymptotic expansions of Fourier transforms and</td>
<td>85</td>
</tr>
<tr>
<td>discrete polyharmonic Green’s functions*</td>
<td></td>
</tr>
<tr>
<td>spaces*</td>
<td></td>
</tr>
<tr>
<td>Walter Philipp, <em>Some metrical theorems in number theory</em></td>
<td>109</td>
</tr>
<tr>
<td>Maxwell Alexander Rosenlicht, *Another proof of a theorem on rational</td>
<td>129</td>
</tr>
<tr>
<td>cross sections*</td>
<td></td>
</tr>
<tr>
<td>Kenneth Allen Ross and Karl Robert Stromberg, *Jessen’s theorem on</td>
<td>135</td>
</tr>
<tr>
<td>Riemann sums for locally compact groups*</td>
<td></td>
</tr>
<tr>
<td>Stephen Simons, *A theorem on lattice ordered groups, results of</td>
<td>149</td>
</tr>
<tr>
<td>Ptak, Namioka and Banach, and a front-ended proof of Lebesgue’s</td>
<td></td>
</tr>
<tr>
<td>theorem*</td>
<td></td>
</tr>
<tr>
<td>Morton Lincoln Slater, <em>On the equation</em> ( \varphi(x) = \int_x^{x+1} K(\xi) f[\varphi(\xi)]d\xi )</td>
<td>155</td>
</tr>
<tr>
<td>Burnett Roland Toskey, *A system of canonical forms for rings on a</td>
<td>179</td>
</tr>
<tr>
<td>direct sum of two infinite cyclic groups*</td>
<td></td>
</tr>
<tr>
<td>Jerry Eugene Vaughan, *A modification of Morita’s characterization of</td>
<td>189</td>
</tr>
<tr>
<td>dimension*</td>
<td></td>
</tr>
</tbody>
</table>