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**A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF  
PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED  
PROOF OF LEBESGUE'S THEOREM**

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# A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED PROOF OF LEBESGUES THEOREM

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The main theorem in this paper is on (not necessarily commutative) lattice ordered groups, and is a generalization of a result on finitely additive set functions due to Namioka. Our result can be used to prove Ptak's combinatorial theorem on convex means, to give a short non measure-theoretic proof of Lebesgue's dominated convergence theorem for a sequence of continuous functions on a countably compact topological space, and to give a short proof of Banach's criteria for the weak convergence of a sequence in the Banach space of all bounded, real functions on an abstract set.

We shall prove the following result.

**THEOREM 1.** *If  $L$  is a lattice ordered group,  $\{g_1, g_2, \dots\}$  is a sequence of positive elements in  $L$ , and  $\varphi$  is an order-preserving homomorphism of  $L$  into the real numbers such that the sequence  $\varphi(g_1 \vee \dots \vee g_p)$  ( $p = 1, 2, \dots$ ) is bounded above and  $\limsup_p \varphi(g_p) > 0$  then there exist integers  $0 < r(1) < r(2) < \dots$  such that, for each  $s$ ,  $\varphi(g_{r(1)} \wedge \dots \wedge g_{r(s)}) > 0$ .*

The idea for this stems from the following result of Namioka.

**Namioka's Theorem.** If  $X$  is a nonvoid set,  $S$  is a field of subsets of  $X$ ,  $\{A_1, A_2, \dots\}$  is a sequence in  $S$  and  $\mu$  is a positive, finitely additive function on  $S$  such that  $\limsup_p \mu(A_p) > 0$  then there exist integers  $0 < r(1) < r(2) < \dots$  such that, for each  $s$ ,  $\mu(A_{r(1)} \cap \dots \cap A_{r(s)}) > 0$ .

Namioka's Theorem can be found in [2, 17.9, p. 157] and [4, Lemma 2, p. 714]—it can clearly be deduced from our result by taking  $L$  to be the set of all  $S$ -simple functions and  $\varphi(\cdot) = \int \cdot d\mu$ . Namioka's theorem was proved in order to give a proof of Krein's Theorem on weak compactness that avoids measure theory. This has also been done, in a superficially very different way, by Ptak, using his combinatorial theorem on the existence of convex means, which appears in print in [3, §24, No. 6, p. 331], [5, 1.3, p. 439] and [6]. Ptak shows that Namioka's theorem can be deduced from his ([5, 5.9, p. 447]—Ptak proves a slightly weaker form in which the conclusion " $\mu(A_{r(1)} \cap \dots \cap A_{r(s)}) > 0$ " is replaced by " $A_{r(1)} \cap \dots \cap A_{r(s)} \neq \emptyset$ ").

**Ptak's Theorem.** We suppose that  $K$  is an infinite set and that  $X$  is a nonvoid family of subsets of  $K$ . We write  $P(K)$  for the collection of all positive, real valued functions  $\lambda$  on  $K$  such that  $\{k: k \in K, \lambda(k) > 0\}$  is finite and  $\sum_{k \in K} \lambda(k) = 1$ ; for  $x \subset K$  we write  $\lambda(x) = \sum_{k \in x} \lambda(k)$ . If

$$\inf_{\lambda \in P(K)} \sup_{x \in X} \lambda(x) > 0$$

then there exist  $x_1, x_2, \dots \in X$  and distinct  $k_1, k_2, \dots \in K$  such that, for each  $s$ ,  $\{k_1, \dots, k_s\} \subset x_s$ .

We shall show that Ptak's Theorem can be deduced from Theorem 1 in a natural way and that the "convexity" is a consequence of the result that weak and the norm closures of a convex subset of a normed linear space coincide.

If  $X$  is a nonvoid set we write  $B(X)$  for the set of all bounded, real functions on  $X$ .  $B(X)$  is a Banach space under the norm  $\|f\| = \sup_{x \in X} |f(x)|$ . We shall show that Theorem 1 can be used to give criteria for the weak convergence of a sequence in  $B(X)$ .

Finally, we shall show how Theorem 1 can be used to give a short non measure-theoretic proof of Lebesgue's dominated convergence theorem for a sequence of *continuous* functions on a countably compact topological space.

Theorem 1 was first proved in a different context. I would like to thank Professor K. Fan for reading the manuscript and suggesting the possibility of an application to Lebesgue's theorem.

**2. Proof of Theorem 1.** Using the identities  $f - f \vee g + g \leq f \wedge g$  and  $(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h) \leq f \wedge h + g \wedge h$ , valid for any positive  $f, g$ , and  $h$  in  $L$ , we see that, if  $q, p$  are integers and  $0 < q < p$ ,

$$\begin{aligned} \varphi(g_p) + \varphi(g_1 \vee \dots \vee g_q) &\leq \varphi((g_1 \vee \dots \vee g_q) \vee g_p) \\ &\quad + \varphi((g_1 \vee \dots \vee g_q) \wedge g_p) \\ &\leq \varphi(g_1 \vee \dots \vee g_p) + \sum_{1 \leq r \leq q} \varphi(g_r \wedge g_p). \end{aligned}$$

Letting  $p \rightarrow \infty$ ,

$$\begin{aligned} \limsup_p \varphi(g_p) + \varphi(g_1 \vee \dots \vee g_q) &\leq \lim_p \varphi(g_1 \vee \dots \vee g_p) \\ &\quad + \sum_{1 \leq r \leq q} \limsup_p \varphi(g_r \wedge g_p). \end{aligned}$$

Letting  $q \rightarrow \infty$ ,

$$\begin{aligned} \limsup_p \varphi(g_p) + \lim_q \varphi(g_1 \vee \dots \vee g_q) &\leq \lim_p \varphi(g_1 \vee \dots \vee g_p) \\ &\quad + \sum_{r \geq 1} \limsup_p \varphi(g_r \wedge g_p). \end{aligned}$$

By hypothesis,  $\lim_p \varphi(g_1 \vee \dots \vee g_p) = \lim_p \varphi(g_1 \vee \dots \vee g_p) < \infty$  and

$\lim \sup_p \varphi(g_p) > 0$ . Hence

$$0 < \sum_{r \geq 1} \lim \sup_p \varphi(g_r \wedge g_p)$$

and thus there exists  $r(1)$  such that  $\lim \sup_p \varphi(g_{r(1)} \wedge g_p) > 0$  — and, from this,  $\varphi(g_{r(1)}) > 0$ .

Performing a similar argument on the elements  $g_{r(1)} \wedge g_{r(1)+1}$ ,  $g_{r(1)} \wedge g_{r(1)+2}$ ,  $\dots$  we can show that there exists  $r(2) > r(1)$  such that  $\lim \sup_p \varphi((g_{r(1)} \wedge g_{r(2)}) \wedge (g_{r(1)} \wedge g_p)) > 0$ , i.e.,

$$\lim \sup_p \varphi(g_{r(1)} \wedge g_{r(2)} \wedge g_p) > 0 \text{ — and, from this } \varphi(g_{r(1)} \wedge g_{r(2)}) > 0 .$$

Continuing this process inductively, we complete the proof of the theorem.

**3. Application of Theorem 1 to Ptak's theorem.** If  $\psi$  is a bounded linear functional on  $B(X)$  then there exists a positive linear functional  $\varphi$  on  $B(X)$  such that, for all  $f \in B(X)$ ,  $|\psi(f)| \leq \varphi(|f|)$ . (If  $f \geq 0$  we define  $\varphi(f)$  to be  $\sup_{|g| \leq f} \psi(g)$  and extend  $\varphi$  to the whole of  $B(X)$  by linearity).

**COROLLARY 2.** *Let  $X$  be a nonvoid set and  $Y$  a bounded subset of  $B(X)$ . Let  $Z$  be the convex extension of  $Y$  in  $B(X)$ . Then (a) implies (b) and (b) implies (c).*

(a)  $\inf_{z \in Z} \|z\| > 0$ .

(b) *There exists a positive linear functional  $\varphi$  on  $B(X)$  such that  $\inf_{y \in Y} \varphi(|y|) > 0$ .*

(c) *For each sequence  $\{y_1, y_2, \dots\}$  in  $Y$  there exist integers  $0 < r(1) < r(2) < \dots$  and  $x_1, x_2, \dots \in X$  such that  $y_{r(t)}(x_s) \neq 0$  whenever  $0 < t \leq s$ .*

*Proof of Corollary 2.* If (a) is true then  $0 \notin$  norm-closure of  $Z$ . Thus, since  $Z$  is convex,  $0 \notin$  weak-closure of  $Z$ , hence  $0 \notin$  weak-closure of  $Y$ . Thus there exists a bounded linear functional  $\psi$  on  $B(X)$  such that  $\inf_{y \in Y} |\psi(y)| > 0$ . If the positive linear functional  $\varphi$  on  $B(X)$  is chosen as in the remarks preceding this Corollary then (b) is satisfied.

If (b) is true,  $\varphi$  is as in (b) and  $y_1, y_2, \dots \in Y$  then  $\lim \sup_p \varphi(|y_p|) > 0$ . We apply Theorem 1 to  $L = B(X)$  and  $g_p = |y_p|$  and find that there exist integers  $0 < r(1) < r(2) < \dots$  such that, for each  $s$ ,  $\varphi(|y_{r(1)}| \wedge \dots \wedge |y_{r(s)}|) > 0$ . It follows from this that (c) is satisfied.

**REMARK.** It can easily be seen that, if  $X, Y$  and  $Z$  are as above and  $\varphi$  is a positive linear functional on  $B(X)$  then

$$\inf_{y \in Y} \varphi(y) \leq \varphi(1) \inf_{z \in Z} \|z\|$$

and so, if all the functions in  $Y$  are *positive*, condition (b) of the Corollary implies condition (a). This is, essentially, the proof used in [5, 5.9, p. 447].

*Proof of Ptak's Theorem.* If  $K$  is any nonvoid set and  $X$  is any nonvoid family of subsets of  $K$  then, for each  $k \in K$ , we define  $y_k \in B(X)$  by  $y_k(x) = 0$  if  $k \notin x$  and  $y_k(x) = 1$  if  $k \in x$ . We apply Corollary 2 ((a) implies (c)) to  $Y = \{y_k: k \in K\}$  and obtain: if

$$\inf_{\lambda \in P(K)} \sup_{x \in X} \lambda(x) > 0$$

then, for all  $k_1, k_2, \dots \in K$ , there exist integers  $0 < r(1) < r(2) < \dots$  and  $x_1, x_2, \dots \in X$  such that, for each  $s$ ,  $\{k_{r(1)}, \dots, k_{r(s)}\} \subset x_s$ . This is formally stronger than, though in fact equivalent to, Ptak's Theorem.

#### 4. Application of Theorem 1 to Lebesgue's theorem of dominated convergence.

DEFINITION 3. Let  $X$  be a nonvoid set and  $f_1, f_2, \dots \in B(X)$ . We shall say that  $\{f_1, f_2, \dots\}$  is a *Dini sequence* if  $|f_1| \wedge \dots \wedge |f_s| \rightarrow 0$  uniformly on  $X$  as  $s \rightarrow \infty$ .

COROLLARY 4. Let  $X$  be a nonvoid set,  $\{f_1, f_2, \dots\}$  be a sequence in  $B(X)$  with the property that all its subsequences are Dini sequences and  $\varphi$  be a positive linear functional on  $B(X)$  such that the sequence  $\varphi(|f_1| \vee \dots \vee |f_p|)$  ( $p = 1, 2, \dots$ ) is bounded above. Then  $\varphi(f_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* Let  $\varepsilon > 0$  be given. For each  $p$  we write  $g_p = (|f_p| - \varepsilon 1) \vee 0$ . If  $0 < r(1) < r(2) < \dots$  are integers then, by hypothesis, for all sufficiently large  $s$ ,  $g_{r(1)} \wedge \dots \wedge g_{r(s)} = 0$  hence

$$\varphi(g_{r(1)} \wedge \dots \wedge g_{r(s)}) = 0.$$

Thus, from Theorem 1,  $\limsup_p \varphi(g_p) = 0$ .

Since  $|f_p| - \varepsilon 1 \leq g_p$ , it follows that  $\limsup_p \varphi(|f_p| - \varepsilon 1) \leq 0$  and hence that  $\limsup_p \varphi(|f_p|) \leq \varepsilon \varphi(1)$ . Since  $\varepsilon$  is arbitrary, this implies that  $\limsup_p \varphi(|f_p|) = 0$ . The required result follows since  $|\varphi(f_n)| \leq \varphi(|f_n|)$ .

If  $X$  is a countably compact topological space we write  $C(X)$  for the set of all real, continuous functions on  $X$ . Corollary 4 is, in fact, true with " $B(X)$ " replaced everywhere by " $C(X)$ "—the proof is identical. In fact any positive linear functional on  $C(X)$  can be extended by the Hahn-Banach Theorem to one on  $B(X)$  and so the result for  $C(X)$  can also be *deduced* from the result for  $B(X)$ .) If  $f_1, f_2, \dots$

$\in C(X)$  and  $f_n \rightarrow 0$  pointwise as  $n \rightarrow \infty$  then, from Dini's Theorem, every subsequence of  $\{f_1, f_2, \dots\}$  is a Dini sequence. The following result is then immediate from the " $C(X)$ " version of Corollary 4.

*Lebesgue's Theorem.* If  $X$  is a countably compact topological space,  $f_1, f_2, \dots \in C(X)$ ,  $f_n \rightarrow 0$  pointwise as  $n \rightarrow \infty$ , and  $\varphi$  is a positive linear functional on  $C(X)$  such that the sequence  $\varphi(|f_1| \vee \dots \vee |f_p|)$  ( $p = 1, 2, \dots$ ) is bounded above, then  $\varphi(f_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Our final result is a slightly expanded form of a theorem due to Banach. (See [1, Annexe, §2, Theorem 5, p. 219] and [5, 5.4, p. 445].)

*Banach's Theorem.* Let  $X$  be a nonvoid set and  $f_1, f_2, \dots$  be a bounded sequence in  $B(X)$ . Then the conditions (a)–(d) are equivalent.

- (a) If  $x_1, x_2, \dots \in X$  then  $\lim_n \lim \inf_i |f_n(x_i)| = 0$ .
- (b) Every subsequence of  $\{f_1, f_2, \dots\}$  is a Dini sequence.
- (c)  $\varphi(|f_n|) \rightarrow 0$  for every positive linear functional  $\varphi$  on  $B(X)$ .
- (d)  $f_n \rightarrow 0$  weakly in  $B(X)$ .

*Proof.* It follows from the definition of a Dini sequence that (a) implies (b), from Corollary 4 that (b) implies (c), and from the remarks preceding Corollary 2 that (c) implies (d).

If (a) is false and  $x_1, x_2, \dots \in X$  are such that

$$\lim \sup_n \lim \inf_i |f_n(x_i)| > \varepsilon > 0$$

then there exist integers  $0 < n(1) < n(2) < \dots$  such that, for each  $k$ ,  $\lim \inf_i |f_{n(k)}(x_i)| > \varepsilon$ . By the diagonal process we can find integers  $0 < i(1) < i(2) < \dots$  such that, for each  $k$ ,  $\lim_j (f_{n(k)})(x_{i(j)})$  exists and has absolute value greater than  $\varepsilon$ . From the Hahn-Banach Theorem, there exists a positive linear functional  $\varphi$  on  $B(X)$  such that  $\varphi(f) = \lim_j f(x_{i(j)})$  whenever  $f \in B(X)$  is such that the limit exists. For this value of  $\varphi$ ,  $\varphi(f_n) \not\rightarrow 0$ . Thus (d) implies (a).

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Leonard Daniel Baumert, <i>Extreme copositive quadratic forms. II</i> .....	1
Edward Lee Bethel, <i>A note on continuous collections of disjoint continua</i> .....	21
Delmar L. Boyer and Adolf G. Mader, <i>A representation theorem for abelian groups with no elements of infinite p-height</i> .....	31
Jean-Claude B. Derderian, <i>Residuated mappings</i> .....	35
Burton I. Fein, <i>Representations of direct products of finite groups</i> .....	45
John Brady Garnett, <i>A topological characterization of Gleason parts</i> .....	59
Herbert Meyer Kamowitz, <i>On operators whose spectrum lies on a circle or a line</i> .....	65
Ignacy I. Kotlarski, <i>On characterizing the gamma and the normal distribution</i> .....	69
Yu-Lee Lee, <i>Topologies with the same class of homeomorphisms</i> .....	77
Moshe Mangad, <i>Asymptotic expansions of Fourier transforms and discrete polyharmonic Green's functions</i> .....	85
Jürg Thomas Marti, <i>On integro-differential equations in Banach spaces</i> ....	99
Walter Philipp, <i>Some metrical theorems in number theory</i> .....	109
Maxwell Alexander Rosenlicht, <i>Another proof of a theorem on rational cross sections</i> .....	129
Kenneth Allen Ross and Karl Robert Stromberg, <i>Jessen's theorem on Riemann sums for locally compact groups</i> .....	135
Stephen Simons, <i>A theorem on lattice ordered groups, results of Ptak, Namioka and Banach, and a front-ended proof of Lebesgue's theorem</i> .....	149
Morton Lincoln Slater, <i>On the equation <math>\varphi(x) = \int_x^{x+1} K(\xi) f[\varphi(\xi)] d\xi</math></i> .....	155
Arthur William John Stoddart, <i>Existence of optimal controls</i> .....	167
Burnett Roland Toskey, <i>A system of canonical forms for rings on a direct sum of two infinite cyclic groups</i> .....	179
Jerry Eugene Vaughan, <i>A modification of Morita's characterization of dimension</i> .....	189