A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED PROOF OF LEBESGUE’S THEOREM

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A THEOREM ON LATTICE ORDERED GROUPS, RESULTS OF PTAK, NAMIOKA AND BANACH, AND A FRONT-ENDED PROOF OF LEBESGUES THEOREM

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The main theorem in this paper is on (not necessarily commutative) lattice ordered groups, and is a generalization of a result on finitely additive set functions due to Namioka. Our result can be used to prove Ptak’s combinatorial theorem on convex means, to give a short non measure-theoretic proof of Lebesgue’s dominated convergence theorem for a sequence of continuous functions on a countably compact topological space, and to give a short proof of Banach’s criteria for the weak convergence of a sequence in the Banach space of all bounded, real functions on an abstract set.

We shall prove the following result.

**Theorem 1.** If $L$ is a lattice ordered group, $\{g_1, g_2, \cdots\}$ is a sequence of positive elements in $L$, and $\varphi$ is an order-preserving homomorphism of $L$ into the real numbers such that the sequence $\varphi(g_1 \vee \cdots \vee g_p)(p = 1, 2, \cdots)$ is bounded above and $\lim \sup_p \varphi(g_p) > 0$ then there exist integers $0 < r(1) < r(2) < \cdots$ such that, for each $s$, $\varphi(g_{r(1)} \wedge \cdots \wedge g_{r(s)}) > 0$.

The idea for this stems from the following result of Namioka.

Namioka’s Theorem. If $X$ is a nonvoid set, $S$ is a field of subsets of $X$, $\{A_1, A_2, \cdots\}$ is a sequence in $S$ and $\mu$ is a positive, finitely additive function on $S$ such that $\lim \sup_p \mu(A_p) > 0$ then there exist integers $0 < r(1) < r(2) < \cdots$ such that, for each $s$, $\mu(A_{r(1)} \cap \cdots \cap A_{r(s)}) > 0$.

Namioka’s Theorem can be found in [2, 17.9, p. 157] and [4, Lemma 2, p. 714]—it can clearly be deduced from our result by taking $L$ to be the set of all $S$-simple functions and $\varphi(\cdot) = \int \cdot \, d\mu$. Namioka’s theorem was proved in order to give a proof of Krein’s Theorem on weak compactness that avoids measure theory. This has also been done, in a superficially very different way, by Ptak, using his combinatorial theorem on the existence of convex means, which appears in print in [3, §24, No. 6, p. 331], [5, 1.3, p. 439] and [6]. Ptak shows that Namioka’s theorem can be deduced from his ([5, 5.9, p. 447]—Ptak proves a slightly weaker form in which the conclusion “$\mu(A_{r(1)} \cap \cdots \cap A_{r(s)}) > 0$” is replaced by “$A_{r(1)} \cap \cdots \cap A_{r(s)} \neq \emptyset$”).
Ptak's Theorem. We suppose that \( K \) is an infinite set and that \( X \) is a nonvoid family of subsets of \( K \). We write \( P(K) \) for the collection of all positive, real valued functions \( \lambda \) on \( K \) such that \( \{k; k \in K, \lambda(k) > 0\} \) is finite and \( \sum_{k \in K} \lambda(k) = 1 \); for \( x \subset K \) we write \( \lambda(x) = \sum_{k \in x} \lambda(k) \). If
\[
\inf_{\lambda \in P(K)} \sup_{x \in X} \lambda(x) > 0
\]
then there exist \( x_1, x_2, \ldots \in X \) and distinct \( k_1, k_2, \ldots \in K \) such that, for each \( s, \{k_1, \ldots, k_s\} \subset x_s \).

We shall show that Ptak's Theorem can be deduced from Theorem 1 in a natural way and that the "convexity" is a consequence of the result that weak and the norm closures of a convex subset of a normed linear space coincide.

If \( X \) is a nonvoid set we write \( B(X) \) for the set of all bounded, real functions on \( X \). \( B(X) \) is a Banach space under the norm \( ||f|| = \sup_{x \in X} |f(x)| \). We shall show that Theorem 1 can be used to give criteria for the weak convergence of a sequence in \( B(X) \).

Finally, we shall show how Theorem 1 can be used to give a short non measure-theoretic proof of Lebesgue's dominated convergence theorem for a sequence of continuous functions on a countably compact topological space.

Theorem 1 was first proved in a different context. I would like to thank Professor K. Fan for reading the manuscript and suggesting the possibility of an application to Lebesgue’s theorem.

2. Proof of Theorem 1. Using the identities \( f - f \lor g + g \leq f \land g \) and \((f \lor g) \land h = (f \land h) \lor (g \land h) \) \leq f \land h + g \land h, valid for any positive \( f, g, \) and \( h \) in \( L \), we see that, if \( q, p \) are integers and \( 0 < q < p \),
\[
\varphi(g_p) + \varphi(g_1 \lor \cdots \lor g_q) \leq \varphi((g_1 \lor \cdots \lor g_q) \lor g_p)
\]
\[
+ \varphi((g_1 \lor \cdots \lor g_q) \land g_p)
\]
\[
\leq \varphi(g_1 \lor \cdots \lor g_p) + \sum_{1 \leq r \leq q} \varphi(g_r \land g_p).
\]

Letting \( p \to \infty \),
\[
\limsup_p \varphi(g_p) + \varphi(g_1 \lor \cdots \lor g_q) \leq \lim_p \varphi(g_1 \lor \cdots \lor g_p)
\]
\[
+ \sum_{1 \leq r \leq q} \limsup_p \varphi(g_r \land g_p).
\]

Letting \( q \to \infty \),
\[
\limsup_p \varphi(g_p) + \lim_q \varphi(g_1 \lor \cdots \lor g_q) \leq \lim_p \varphi(g_1 \lor \cdots \lor g_p)
\]
\[
+ \sum_{r \geq 1} \limsup_p \varphi(g_r \land g_p).
\]

By hypothesis, \( \lim_p \varphi(g_1 \lor \cdots \lor g_q) = \lim_p \varphi(g_1 \lor \cdots \lor g_p) < \infty \) and
lim sup_p φ(g_p) > 0. Hence
\[ 0 < \sum_{r \geq 1} \lim sup_p \varphi(g_r \land g_p) \]
and thus there exists \( r(1) \) such that \( \lim sup_p \varphi(g_{r(1)} \land g_p) > 0 \) — and, from this, \( \varphi(g_{r(1)}) > 0 \).

Performing a similar argument on the elements \( g_{r(1)} \land g_{r(1)+1}, g_{r(1)} \land g_{r(1)+2}, \cdots \) we can show that there exists \( r(2) > r(1) \) such that
\[ \lim sup_p \varphi((g_{r(1)} \land g_{r(2)}) \land (g_{r(1)} \land g_p)) > 0, \]
i.e.,
\[ \lim sup_p \varphi(g_{r(1)} \land g_{r(2)} \land g_p) > 0 \] — and, from this \( \varphi(g_{r(1)} \land g_{r(2)}) > 0 \).

Continuing this process inductively, we complete the proof of the theorem.

3. Application of Theorem 1 to Ptak’s theorem. If \( \psi \) is a bounded linear functional on \( B(X) \) then there exists a positive linear functional \( \varphi \) on \( B(X) \) such that, for all \( f \in B(X), | \psi(f) | \leq \varphi(|f|) \). (If \( f \geq 0 \) we define \( \varphi(f) \) to be \( \sup_{|g| \leq f} \psi(g) \) and extend \( \varphi \) to the whole of \( B(X) \) by linearity).

COROLLARY 2. Let \( X \) be a nonvoid set and \( Y \) a bounded subset of \( B(X) \). Let \( Z \) be the convex extension of \( Y \) in \( B(X) \). Then (a) implies (b) and (b) implies (c).

(a) \( \inf_{y \in Y} \| y \| > 0 \).

(b) There exists a positive linear functional \( \varphi \) on \( B(X) \) such that \( \inf_{y \in Y} \varphi(\| y \|) > 0 \).

(c) For each sequence \( \{y_1, y_2, \cdots\} \) in \( Y \) there exist integers \( 0 < r(1) < r(2) < \cdots \) and \( x_1, x_2, \cdots \in X \) such that \( y_{r(1)}(x_s) \neq 0 \) whenever \( 0 < t \leq s \).

Proof of Corollary 2. If (a) is true then \( 0 \in \text{norm-closure of } Z \). Thus, since \( Z \) is convex, \( 0 \in \text{weak-closure of } Z \), hence \( 0 \in \text{weak-closure of } Y \). Thus there exists a bounded linear functional \( \psi \) on \( B(X) \) such that \( \inf_{y \in Y} | \psi(y) | > 0 \). If the positive linear functional \( \varphi \) on \( B(X) \) is chosen as in the remarks preceding this Corollary then (b) is satisfied.

If (b) is true, \( \varphi \) is as in (b) and \( y_1, y_2, \cdots \in Y \) then \( \lim sup_p \varphi(\| y_p \|) > 0 \). We apply Theorem 1 to \( L = B(X) \) and \( g_p = \| y_p \| \) and find that there exist integers \( 0 < r(1) < r(2) < \cdots \) such that, for each \( s, \varphi(\| y_{r(1)} \| \land \cdots \land \| y_{r(s)} \|) > 0 \). It follows from this that (c) is satisfied.

REMARK. It can easily be seen that, if \( X, Y \) and \( Z \) are as above and \( \varphi \) is a positive linear functional on \( B(X) \) then
and so, if all the functions in \( Y \) are positive, condition (b) of the Corollary implies condition (a). This is, essentially, the proof used in [5, 5.9, p. 447].

**Proof of Ptak's Theorem.** If \( K \) is any nonvoid set and \( X \) is any nonvoid family of subsets of \( K \) then, for each \( k \in K \), we define \( y_k \in B(X) \) by \( y_k(x) = 0 \) if \( k \not\in x \) and \( y_k(x) = 1 \) if \( k \in x \). We apply Corollary 2 ((a) implies (c)) to \( Y = \{y_k: k \in K\} \) and obtain: if

\[
\inf_{\lambda \in \mathcal{P}(K)} \sup_{x \in X} \lambda(x) > 0
\]

then, for all \( k_1, k_2, \cdots \in K \), there exist integers \( 0 < r(1) < r(2) < \cdots \) and \( x_1, x_2, \cdots \in X \) such that, for each \( s \), \( \{k_{r(1)}, \cdots, k_{r(s)}\} \subseteq x_s \). This is formally stronger than, though in fact equivalent to, Ptak's Theorem.

4. Application of Theorem 1 to Lebesgue's theorem of dominated convergence.

**Definition 3.** Let \( X \) be a nonvoid set and \( f_1, f_2, \cdots \in B(X) \). We shall say that \( \{f_1, f_2, \cdots\} \) is a Dini sequence if \( |f_1| \wedge \cdots \wedge |f_s| \to 0 \) uniformly on \( X \) as \( s \to \infty \).

**Corollary 4.** Let \( X \) be a nonvoid set, \( \{f_1, f_2, \cdots\} \) be a sequence in \( B(X) \) with the property that all its subsequences are Dini sequences and \( \varphi \) be a positive linear functional on \( B(X) \) such that the sequence \( \varphi(|f_1| \vee \cdots \vee |f_p|) \) \( (p = 1, 2, \cdots) \) is bounded above. Then \( \varphi(f_n) \to 0 \) as \( n \to \infty \).

**Proof.** Let \( \varepsilon > 0 \) be given. For each \( p \) we write \( g_p = (|f_p| - \varepsilon 1) \vee 0 \). If \( 0 < r(1) < r(2) < \cdots \) are integers then, by hypothesis, for all sufficiently large \( s \), \( g_{r(1)} \wedge \cdots \wedge g_{r(s)} = 0 \) hence

\[
\varphi(g_{r(1)} \wedge \cdots \wedge g_{r(s)}) = 0.
\]

Thus, from Theorem 1, \( \lim \sup_p \varphi(g_p) = 0 \).

Since \( |f_p| - \varepsilon 1 \leq g_p \), it follows that \( \lim \sup_p \varphi(|f_p| - \varepsilon 1) \leq 0 \) and hence that \( \lim \sup_p \varphi(|f_p|) \leq \varepsilon \varphi(1) \). Since \( \varepsilon \) is arbitrary, this implies that \( \lim \sup_p \varphi(|f_n|) = 0 \). The required result follows since \( |\varphi(f_n)| \leq \varphi(|f_n|) \).

If \( X \) is a countably compact topological space we write \( C(X) \) for the set of all real, continuous functions on \( X \). Corollary 4 is, in fact, true with "\( B(X) \)" replaced everywhere by "\( C(X) \)"—the proof is identical. In fact any positive linear functional on \( C(X) \) can be extended by the Hahn-Banach Theorem to one on \( B(X) \) and so the result for \( C(X) \) can also be deduced from the result for \( B(X) \).
e \in C(X) and \( f_n \to 0 \) pointwise as \( n \to \infty \) then, from Dini's Theorem, every subsequence of \( \{f_1, f_2, \ldots\} \) is a Dini sequence. The following result is then immediate from the "C(X)" version of Corollary 4.

**Lebesgue's Theorem.** If \( X \) is a countably compact topological space, \( f_1, f_2, \ldots \in C(X) \), \( f_n \to 0 \) pointwise as \( n \to \infty \), and \( \varphi \) is a positive linear functional on \( C(X) \) such that the sequence \( \varphi(|f_1| \vee \cdots \vee |f_p|)(p = 1, 2, \ldots) \) is bounded above, then \( \varphi(f_n) \to 0 \) as \( n \to \infty \).

Our final result is a slightly expanded form of a theorem due to Banach. (See [1, Annexe, §2, Theorem 5, p. 219] and [5, 5.4, p. 445].)

**Banach's Theorem.** Let \( X \) be a nonvoid set and \( f_1, f_2, \ldots \) be a bounded sequence in \( B(X) \). Then the conditions (a)—(d) are equivalent.

(a) If \( x_1, x_2, \ldots \in X \) then \( \lim_n \liminf_i |f_n(x_i)| = 0 \).

(b) Every subsequence of \( \{f_1, f_2, \ldots\} \) is a Dini sequence.

(c) \( \varphi(|f_n|) \to 0 \) for every positive linear functional \( \varphi \) on \( B(X) \).

(d) \( f_n \to 0 \) weakly in \( B(X) \).

**Proof.** If follows from the definition of a Dini sequence that (a) implies (b), from Corollary 4 that (b) implies (c), and from the remarks preceding Corollary 2 that (c) implies (d).

If (a) is false and \( x_1, x_2, \ldots \in X \) are such that

\[ \limsup_n \liminf_i |f_n(x_i)| > \varepsilon > 0 \]

then there exist integers \( 0 < n(1) < n(2) < \cdots \) such that, for each \( k \), \( \liminf_i |f_{n(k)}(x_i)| > \varepsilon \). By the diagonal process we can find integers \( 0 < i(1) < i(2) < \cdots \) such that, for each \( k \), \( \lim_j (f_{n(i(k))}(x_{i(k)})) \) exists and has absolute value greater than \( \varepsilon \). From the Hahn-Banach Theorem, there exists a positive linear functional \( \varphi \) on \( B(X) \) such that \( \varphi(f) = \lim_j f(x_{i(j)}) \) whenever \( f \in B(X) \) is such that the limit exists. For this value of \( \varphi \), \( \varphi(f_n) \to 0 \). Thus (d) implies (a).

**References**


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