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**EXISTENCE OF OPTIMAL CONTROLS**

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## EXISTENCE OF OPTIMAL CONTROLS

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Let  $f = (f_1, f_2, \dots, f_n)$  be a mapping to  $E_n$  from a set  $D$  in  $E_1 \times E_n \times E_m$ ; and  $f_0$  a real function on  $D$ . Consider a "control" function  $u$  from an interval  $I = [t_0, t_1]$  in  $E_1$  to  $E_m$ ; and a "response" function  $x$  from  $I$  to  $E_n$  such that  $(t, x(t), u(t)) \in D$  for almost every  $t \in I$ ,  $f_0(t, x(t), u(t))$  has an integral (finite or  $+\infty$ ) on  $I$ ,  $f(t, x(t), u(t))$  is integrable on  $I$ , and

$$(1) \quad x(t) = x(t_0) + \int_{t_0}^t f(s, x(s), u(s)) ds$$

for all  $t \in I$ . In a class  $\Gamma$  of such control-response pairs  $(u, x)$ , a pair  $(u^*, x^*)$  is called optimal (with respect to  $f_0$ ) if the "cost" functional

$$C(u, x) = (I) \int f_0(t, x, u) dt$$

has a minimum at  $(u^*, x^*)$ . Here we consider conditions sufficient for existence of such optimal pairs.

The problem of existence of optimal controls for various functions  $f, f_0$  and classes  $\Gamma$  has been treated in [6], [11], [7], [5], [8], [9], [13], [10], [1], [2], and [3]. Gamkrelidze [6] assumed  $f_0$  constant,  $f$  linear in  $(x, u)$ , and  $u$  restricted to a cube  $U$  in  $E_m$ . Pontryagin [11] extended Gamkrelidze's work to the situation where  $U$  is any compact convex polyhedron. Lee and Markus [8] considered  $f$  and  $f_0$  linear in  $u$ , and  $U$  any compact convex set. Simple integral restraints on  $u$  were treated by Krasovskii [7] and Neustadt [9].

The conditions on  $U$  and  $f$  for  $f_0$  constant were relaxed remarkably by Filippov [5], who considered a variable compact restraint set  $U(t, x)$  such that the set  $f(t, x, U(t, x))$  is convex for each  $(t, x)$ . Roxin [13], in effect, considered  $U$  a fixed compact set with  $(f, f_0)(t, x, U)$  convex. By taking  $f$  and  $f_0$  linear in  $x$  and  $U$  compact, Neustadt [10] avoided all convexity assumptions. Cesari [1] assumes  $U(t, x)$  compact,  $f(t, x, U(t, x))$  convex, and  $f_0$  sufficiently convex in  $u$  compared with the curvature of  $f$  in  $u$ . In [3], Cesari extends considerations to restraint sets  $U(t, x)$  which can be unbounded.

In this paper, we consider variations of the conditions above for the case in which  $f$  is linear in  $u$ ,  $f_0$  is convex in  $u$ , and the variable restraint set  $U(t, x)$  is convex and closed but not necessarily bounded. In particular, integral restraints are taken into account, and used as an alternative source for the fundamental compactness condition. In a later section, we apply our results to classical existence problems of the calculus of variations.

2. **Definitions.** We shall call a real function  $\phi(t, x, u)$  "linearly bounded below in  $u$ " if

$$\phi(t, x, u) \geq p(t, x) + u \cdot q(t, x)$$

for some uniformly continuous and bounded functions  $p, q$ . The meaning of "linearly bounded in  $u$ " will be obvious.

Consider the following sets, functions, and numbers.

(2) The sets  $J_0 = [T_0, T'_0]$ ,  $J_1 = [T_1, T'_1]$  are compact intervals in  $E_1$  with  $T_0 \leq T'_1$ . Let  $J = [T_0, T'_1]$ .

(3) The set  $B$  is a closed set in  $J \times E_n$ , and  $U$  is a closed convex set in  $E_m$ . Let  $D = B \times U$ .

(4) The real continuous functions  $h_j(t, x, u)$  on  $D$ , at most countable in number, are convex and linearly bounded below in  $u$ . Let  $U(t, x) = U \cap \{u: h_j(t, x, u) \leq 0 \text{ for all } j\}$ .

(5) The mapping  $G_0(t)$ , from  $J_0$  to the class of compact sets in  $E_n$ , is continuous in the Hausdorff sense. The mapping  $G_1(t)$ , from  $J_1$  to the class of closed sets in  $E_n$ , is also continuous in the Hausdorff sense.

(6) The real continuous functions  $g_k(t, x, u)$  on  $D$  are convex and linearly bounded below in  $u$ ;  $c_k$  are corresponding real numbers.

(7) The continuous mapping  $f(t, x, u)$  from  $D$  to  $E_n$  is linear in  $u$  and with each component function  $f_i$  linearly bounded in  $u$ . Note that linear bounding of each  $f_i$  does not follow from linearity, even if the coefficients in  $f$  are bounded; for example,  $f = u \sin x^2$  on  $E_1 \times E_1$ . However, if the coefficients in  $f$  are bounded and each component of  $u$  in  $U$  is bounded above or below (in particular,  $U$  bounded), then linearity implies linear bounding.

Define  $\Gamma$  to be the class of all control-response pairs  $(u, x)$  on intervals  $I = [t_0, t_1]$ , such that (1) holds, and

$$(8) \quad t_0 \in J_0, \quad t_1 \in J_1;$$

$$(9) \quad (t, x(t)) \in B \quad \text{for every } t \in I;$$

$$(10) \quad x(t_0) \in G_0(t_0), \quad x(t_1) \in G_1(t_1);$$

$$(11) \quad u(t) \in U(t, x(t)) \quad \text{for almost every } t \in I;$$

$$(12) \quad (I) \int g_k(t, x, u) dt \leq c_k \quad \text{for each } k.$$

We shall assume that

(13)  $\int |u|$  is equi absolutely continuous on  $I$ ;

that is, for any  $\varepsilon > 0$ , there exists  $\lambda(\varepsilon) > 0$  such that  $(M) \int |u| dt < \varepsilon$  for any  $(u, x) \in \Gamma$  and measurable set  $M \subseteq I$  for which the Lebesgue measure  $\mu(M) < \lambda(\varepsilon)$ . (Conditions sufficient for this will be discussed in § 6.) Note that  $(I) \int |u| dt$  is then bounded on  $I$ .

Note also that, without further restrictions on  $f$ ,  $x$  is not necessarily determined through (1) by  $x(t_0)$  and  $u$ .

Our general approach will be to prove that the class  $\Gamma$  is sequentially compact and closed in an appropriate convergence system. We then apply a general semicontinuity theorem of [14] to obtain the existence of a minimum for  $C(u, x)$  on  $\Gamma$ .

**3. A compactness theorem.** We first prove a compactness theorem for  $\Gamma$ . It is essentially an abstraction of techniques of Tonelli [15] and Lee and Markus [8].

**THEOREM 1.** *Any infinite subclass of  $\Gamma$  contains a sequence  $(u^n, x^n)$  such that there exist a compact interval  $I^* = [t_0^*, t_1^*]$ , a continuous mapping  $x^*$  from  $I^*$  to  $E_n$ , and an integrable mapping  $u^*$  from  $I^*$  to  $E_m$ , for which*

- (14) (a)  $t_0^n \rightarrow t_0^*, t_1^n \rightarrow t_1^*$ ;  
 (b)  $x^n(t_0^n) \rightarrow x^*(t_0^*), x^n(t_1^n) \rightarrow x^*(t_1^*)$ ;  
 (c)  $\sup \{|x^n(t) - x^*(t)| : t \in I^n \cap I^*\} \rightarrow 0$ ; and  
 (d)  $(I^n \cap E) \int u^n dt \rightarrow (I^* \cap E) \int u^* dt$

for every measurable set  $E \subseteq E_1$ .

*Proof.* The linear bounding of the component functions  $f_i$  gives  $|f| \leq a + b|u|$  for some constants  $a, b$ . Then

$$|x(t') - x(t)| \leq a(t' - t) + b \int_t^{t'} |u| ds$$

from (1); thus  $x$  is equicontinuous on  $\Gamma$ .

All  $G_0(t)$  and  $J_0$  are compact; hence, by an elementary argument,  $\bigcup G_0(t)$  is compact. In addition,  $x$  is equicontinuous and  $J$  is bounded; hence  $x$  is equibounded on  $\Gamma$ .

On  $J$ , define

$$\begin{aligned} x_+(t) &= x(t_0) && \text{on } [T_0, t_0], \\ & x(t) && \text{on } [t_0, t_1], \\ & x(t_1) && \text{on } [t_1, T_1']; \\ u_+(t) &= 0 && \text{on } [T_0, t_0) \text{ and } (t_1, T_1'], \\ & u(t) && \text{on } [t_0, t_1]. \end{aligned}$$

Let  $\Gamma_+$  be the corresponding class of pairs  $(u_+, x_+)$ . On  $\Gamma_+$ ,  $x_+$  is equicontinuous and equibounded, and  $\int |u_+|$  is equi absolutely continuous and  $(J)\int |u_+| dt$  is bounded. Consequently, from any infinite subclass of  $\Gamma_+$ , we can extract in succession sequences  $t_0^n \rightarrow t_0^*$ ,  $t_1^n \rightarrow t_1^*$ ,  $x_+^n \rightarrow x_+^*$  uniformly on  $J$ , and  $u_+^n \rightarrow u_+^*$  weakly in  $L_1(J)$  [4, p. 294] for some  $t_0^*$ ,  $t_1^*$ , continuous  $x_+^*$ , and integrable  $u_+^*$ .

Define  $x^* = x_+^* | [t_0^*, t_1^*]$ ,  $u^* = u_+^* | [t_0^*, t_1^*]$ . Then  $x^*$  is continuous and  $u^*$  is integrable. Since  $x_+^n(t_0^*) \rightarrow x_+^*(t_0^*)$ ,  $x_+^n$  is equicontinuous, and  $t_0^n \rightarrow t_0^*$ , we have  $x^n(t_0^n) \rightarrow x^*(t_0^*)$ . Similarly,  $x^n(t_1^n) \rightarrow x^*(t_1^*)$ .

For any  $\varepsilon > 0$ , there exists  $\lambda(\varepsilon) > 0$  such that  $(E)\int |u_+| dt < \varepsilon$  for any set  $E \subseteq J$  with Lebesgue measure  $\mu(E) < \lambda(\varepsilon)$ . Now

$$(J)\int \phi u_+^n dt \rightarrow (J)\int \phi u_+^* dt$$

for every  $\phi \in L_\infty(J)$ . For any measurable set  $E \subseteq E_1$ , take  $\phi$  as the characteristic function of  $I^* \cap E$ . Then

$$\left| (I^* \cap E)\int u_+^n dt - (I^* \cap E)\int u_+^* dt \right| < \varepsilon$$

for  $n$  greater than some  $N(\varepsilon, E)$ . Now

$$\begin{aligned} \mu(I^* \cap E - I^n \cap E) + \mu(I^n \cap E - I^* \cap E) \\ \leq |t_0^n - t_0^*| + |t_1^n - t_1^*|, \end{aligned}$$

which is less than  $\lambda(\varepsilon)$  for  $n$  greater than some  $N(\varepsilon)$ . Hence, for  $n > N(\varepsilon)$  and  $N(\varepsilon, E)$ ,

$$\left| (I^n \cap E)\int u^n dt - (I^* \cap E)\int u^* dt \right| < 2\varepsilon.$$

**4: Continuity and semicontinuity.** The following continuity theorem is required for the semicontinuity theorem.

**THEOREM 2.** *Let  $(u^n, x^n)$  be a sequence in  $\Gamma$  converging to  $(u^*, x^*)$  in the sense (14). Let the functions  $p: B \rightarrow E_1$  and  $q: B \rightarrow E_m$  be uniformly continuous and bounded. Then, for every measurable set  $E \subseteq E_1$ ,*

$$\begin{aligned} (I^n \cap E)\int [p(t, x^n) + u^n \cdot q(t, x^n)] dt \\ \rightarrow (I^* \cap E)\int [p(t, x^*) + u^* \cdot q(t, x^*)] dt. \end{aligned}$$

*Proof.* Note that, since  $B$  is closed, conditions (14) (a), (b), (c) ensure that  $(t, x^*(t)) \in B$  for every  $t \in I^*$ .

We express our conditions in explicit form. For any  $\varepsilon > 0$ , there exists  $N(\varepsilon)$  such that, for  $n > N(\varepsilon)$ ,

$$\begin{aligned} |t_0^n - t_0^*| &< \varepsilon, \\ |t_1^n - t_1^*| &< \varepsilon, \\ |x^n(t_0^n) - x^*(t_0^*)| &< \varepsilon, \\ |x^n(t_1^n) - x^*(t_1^*)| &< \varepsilon, \end{aligned}$$

and

$$|x^n(t) - x^*(t)| < \varepsilon \quad \text{if } t \in I^n \cap I^* ;$$

and, for any measurable set  $E \subseteq E_i$ , there exists  $N(\varepsilon, E)$  such that, for  $n > N(\varepsilon, E)$ ,

$$\left| (I^n \cap E) \int u^n dt - (I^* \cap E) \int u^* dt \right| < \varepsilon .$$

Also, there exists  $\lambda(\varepsilon) > 0$  such that  $(M) \int |u^n| dt < \varepsilon$  for all  $n$  and for every measurable set  $M \subseteq I^n$  with  $\mu(M) < \lambda(\varepsilon)$ ; and there exists  $\alpha$  such that  $(I^n) \int |u^n| dt < \alpha$  for all  $n$ . In addition, there exists  $\delta(\varepsilon) > 0$  such that  $|p(t, x) - p(t', x')|$  and  $|q(t, x) - q(t', x')| < \varepsilon$  for  $|t - t'|$  and  $|x - x'| < \delta(\varepsilon)$ ; and there exists  $\beta$  such that  $|p(t, x)|$  and  $|q(t, x)| < \beta$  for all  $(t, x) \in B$ .

Now

$$\begin{aligned} &\left| (I^n \cap E) \int p(t, x^n) dt - (I^* \cap E) \int p(t, x^*) dt \right| \\ &\leq \beta(|t_1^n - t_1^*| + |t_0^n - t_0^*|) \\ &\quad + (I^n \cap E \cap I^*) \int |p(t, x^n) - p(t, x^*)| dt \\ &< \beta\varepsilon + (T_1^n - T_0)\varepsilon \quad \text{if } n > N\left(\frac{1}{2}\varepsilon\right) \text{ and } N(\delta(\varepsilon)) . \end{aligned}$$

Also,

$$\begin{aligned} &\left| (I^n \cap E) \int u^n \cdot q(t, x^n) dt - (I^n \cap E \cap I^*) \int u^n \cdot q(t, x^*) dt \right| \\ &\leq \beta(I^n - I^*) \int |u^n| dt \\ &\quad + \left| (I^n \cap E \cap I^*) \int u^n \cdot [q(t, x^n) - q(t, x^*)] dt \right| \\ &< \beta\varepsilon + \alpha\varepsilon \quad \text{if } n > N\left(\frac{1}{2}\lambda(\varepsilon)\right) \text{ and } N(\delta(\varepsilon)) . \end{aligned}$$

By uniform continuity of  $x^*$  on  $I^*$ , there exists  $\gamma(\varepsilon) > 0$  such that  $|x^*(t) - x^*(t')| < \varepsilon$  for  $|t - t'| < \gamma(\varepsilon)$ . Divide  $I^*$  into  $\sigma$  intervals  $I_\sigma$

with lengths less than  $\delta(\varepsilon)$  and  $\gamma(\delta(\varepsilon))$ , and take  $q_s = q(t, x^*(t))$  for some  $t \in I_s$ . Then

$$\begin{aligned} & \left| (I^n \cap E \cap I^*) \int u^n \cdot q(t, x^*) dt - (I^* \cap E) \int u^* \cdot q(t, x^*) dt \right| \\ &= \left| \sum (I^n \cap E \cap I_s) \int u^n \cdot q(t, x^*) dt - \sum (I_s \cap E) \int u^* \cdot q(t, x^*) dt \right| \\ &< \left| \sum q_s \cdot \left[ (I^n \cap E \cap I_s) \int u^n dt - (I_s \cap E) \int u^* dt \right] \right| \\ &\quad + \varepsilon (I^*) \int |u^*| dt + \alpha \varepsilon \\ &< \varepsilon + \varepsilon (I^*) \int |u^*| dt + \alpha \varepsilon \quad \text{if } n > \max N(\varepsilon/\sigma | q_s |, I_s \cap E). \end{aligned}$$

We shall use repeatedly the following semicontinuity theorem.

**THEOREM 3.** *Let  $\phi(t, x, u)$  be a real continuous function on  $D$ , convex and linearly bounded below in  $u$ . Consider a sequence  $(u^n, x^n) \in \Gamma$  converging to  $(u^*, x^*)$  in the sense (14). (We shall prove in § 5 that  $u^*(t) \in U$  for almost every  $t \in I^*$ ; and  $(t, x^*(t)) \in B$  for every  $t \in I^*$ .) Then, for every measurable set  $E \subseteq E_1$ ,*

$$(I^* \cap E) \int \phi(t, x^*, u^*) dt \leq \liminf (I^n \cap E) \int \phi(t, x^n, u^n) dt.$$

Theorem 3 follows easily from Theorem 4 of [14]. Our convergence (14) satisfies condition (10) of [14]. The discussion of § 6 of [14] applies here, since the lower bound integral is continuous.

## 5. A closure theorem.

**THEOREM 4.** *Let  $(u^n, x^n)$  be a sequence in  $\Gamma$  converging to  $(u^*, x^*)$  in the sense (14). Then  $(u^*, x^*) \in \Gamma$ .*

*Proof.* By (14a),  $t_0^* \in J_0$  and  $t_1^* \in J_1$ .

For  $t_0^* < t < t_1^*$ ,  $t \in I^n$  for all sufficiently large  $n$ , so  $x^n(t) \rightarrow x^*(t)$  by (14c). Thus  $(t, x^*(t)) \in B$ . In addition, (14a) and (14b) give  $(t_0^*, x^*(t_0^*))$  and  $(t_1^*, x^*(t_1^*)) \in B$ .

If  $x^*(t_0^*)$  were not in  $G_0(t_0^*)$ , then it would not be in the closure  $N^c$  of some neighbourhood  $N$  of  $G_0(t_0^*)$ . But  $G_0(t) \subseteq N$  for  $t$  sufficiently near  $t_0^*$ , so  $x^n(t_0^*) \in N$  for all sufficiently large  $n$ , from which  $x^*(t_0^*) \in N^c$ ! Similarly,  $x^*(t_1^*) \in G_1(t_1^*)$ .

The closed convex set  $U$  in  $E_m$  is the intersection of a countable number of half spaces  $\{u: \beta + u \cdot b \leq 0\}$ . Let

$$E = \{t: t \in I^*, \beta + u^*(t) \cdot b > 0\}.$$

Then

$$0 \geq (I^n \cap E) \int (\beta + u^n \cdot b) dt \rightarrow (E) \int (\beta + u^* \cdot b) dt \geq 0 .$$

Thus  $\mu(E) = 0$ , and so  $u^*(t) \in U$  for almost every  $t \in I^*$ .

Let  $E_j = \{t: t \in I^*, h_j(t, x^*(t), u^*(t)) > 0\}$ . Now

$$(I^n \cap E_j) \int h_j(t, x^n, u^n) dt \leq 0 ,$$

so

$$(E_j) \int h_j(t, x^*, u^*) dt \leq 0$$

by Theorem 3. Consequently,  $\mu(E_j) = 0$ . Thus  $(u^*, x^*)$  satisfies (11).

By Theorem 3 with  $E = E_1$ , the integral  $(I) \int g_k(t, x, u) dt$  is lower semicontinuous in the convergence (14). Consequently

$$(I^*) \int g_k(t, x^*, u^*) dt \leq c_k ,$$

that is, condition (12) is satisfied.

Consider  $t$  such that  $t_0^* < t < t_1^*$ . Theorem 3 with  $E = \{s: s \leq t\}$  shows that the integral

$$\int_{t_0}^t f(s, x(s), u(s)) ds$$

is continuous in the convergence (14). Also,  $x^n(t) \rightarrow x^*(t)$  and  $x^n(t_0^n) \rightarrow x^*(t_0^*)$ . Thus condition (1) on  $(u^n, x^n)$  carries over to  $(u^*, x^*)$ . For  $t = t_1^*$ , a similar argument applies, but with  $E = E_1$ .

Thus  $(u^*, x^*)$  satisfies conditions (1) and (8) through (12).

## 6. The existence theorem.

**THEOREM 5.** *Let the real continuous function  $f_0(t, x, u)$  on  $D$  be convex and linearly bounded below in  $u$ . Assume, as previously, that  $\Gamma$  satisfies condition (13). Then, if  $\Gamma$  is not empty,  $C(u, x)$  has a minimum on  $\Gamma$ .*

*Proof.* Theorems 1 and 4 show that  $\Gamma$  is sequentially compact in itself with respect to the convergence (14).

Since  $f_0$  is linearly bounded below in  $u$  and  $u$  is integrable,  $f_0(t, x, u)$  has an integral, finite or  $+\infty$ . Theorem 3, with  $E = E_1$ , shows that  $(I) \int f_0(t, x, u) dt$  is lower semicontinuous with respect to the convergence (14).

A lower semicontinuous functional on a sequentially compact space has a minimum. Hence the result.

7. **Equi absolute continuity of  $\int |u|$ .** Condition (13) plays the key part in our compactness theorem. We now study conditions sufficient for equi absolute continuity of  $\int |u|$  on  $\Gamma$ .

For example, if the set  $U$  and the functions  $h_j$  are such that

$$U(t, x) = U \cap \{u: h_j(t, x, u) \leq 0 \text{ for all } j\}$$

is bounded uniformly on  $B$ , then condition (13) is obviously satisfied. This is the standard situation in problems of optimal control.

The following more general integral condition is quite standard in the calculus of variations.

**THEOREM 6.** *Let  $\psi(u)$  be a real function on  $E_m$ , bounded below and such that  $\psi(u)/|u| \rightarrow \infty$  as  $|u| \rightarrow \infty$ . If  $(I) \int \psi(u(t)) dt$  is bounded on  $\Gamma$ , then  $\int |u|$  is equi absolutely continuous.*

*Proof.* Suppose that  $(I) \int \psi(u) dt \leq c$  on  $\Gamma$ ;  $\psi(u) \geq b$ ; and, for any  $\varepsilon > 0$ ,  $(\psi(u) - b)/|u| > 1/\varepsilon$  for  $|u| > m(\varepsilon)$ . For any  $(u, x) \in \Gamma$  and measurable set  $M \subseteq I$ , define

$$M^+ = M \cap \{t: t \in I, |u(t)| > m(\varepsilon)\}, \quad M^- = M - M^+.$$

Then

$$\begin{aligned} (M) \int |u(t)| dt &= (M^+) \int |u| dt + (M^-) \int |u| dt \\ &\leq \varepsilon (M^+) \int (\psi(u) + |b|) dt + m(\varepsilon) \mu(M^-) \\ &\leq \varepsilon (I) \int (\psi(u) + |b|) dt + m(\varepsilon) \mu(M) \\ &< \varepsilon (c + |b| (T_1' - T_0) + 1) \end{aligned}$$

if  $\mu(M) < \varepsilon/m(\varepsilon)$ . Thus  $\int |u|$  is equi absolutely continuous.

For example, a "growth condition"  $g_k(t, x, u) \geq \psi(u)$  on some  $g_k$  would be sufficient for the bounding of  $(I) \int \psi(u) dt$  on  $\Gamma$ . Alternatively, the bounding of  $(I) \int \psi(u) dt$ , sufficiently for our purpose, would follow from a similar growth condition on  $f_0$ .

**THEOREM 7.** *Suppose that  $f_0(t, x, u) \geq \psi(u)$ , where  $\psi$  has the properties stated in Theorem 6. Then our existence theorem, Theorem 5, holds without the direct assumption of condition (13).*

*Proof.* If  $C(u, x) = \infty$  for all  $(u, x) \in \Gamma$ , then the result is trivial. Otherwise, there exists  $(u_1, x_1) \in \Gamma$  with  $C(u_1, x_1) < \infty$ . In considering a minimum for  $C(u, x)$  on  $\Gamma$ , we can restrict consideration to the class

$$\Gamma_1 = \Gamma \cap \{(u, x) : C(u, x) \leq C(u_1, x_1)\}.$$

Then  $(I) \int \psi(u) dt$  is bounded on  $\Gamma_1$ . Theorems 5 and 6 show that  $C(u, x)$  has a minimum on  $\Gamma_1$ , which is obviously also a minimum on  $\Gamma$ .

**8. Extension to unbounded intervals  $J_0, J_1$ .** If  $J_1$  has semi-infinite form, then our existence theorem still holds, provided  $f_0$  is positively bounded below.

**THEOREM 8.** *Assume that  $f_0(t, x, u) \geq m$  for some positive constant  $m$ . Then Theorem 5 holds also for  $J_1$  of the form  $[T_1, \infty)$ .*

*Proof.* If  $C(u, x) = \infty$  for all  $(u, x) \in \Gamma$ , then the existence of a minimum for  $C(u, x)$  is trivial. Otherwise,  $C_1 = C(u_1, x_1) < \infty$  for some  $(u_1, x_1) \in \Gamma$ . We can restrict consideration to the class  $\Gamma_1$  of those  $(u, x) \in \Gamma$  for which  $C(u, x) \leq C_1$ .

For  $(u, x) \in \Gamma_1$ ,  $C_1 \geq C(u, x) \geq m(t_1 - t_0)$ , so

$$t_1 \leq t_0 + C_1/m \leq T'_0 + C_1/m.$$

Thus the condition  $t_1 \in [T_1, T'_0 + C_1/m]$  does not further restrict  $\Gamma_1$ . Then Theorem 5 shows that  $C(u, x)$  has a minimum on  $\Gamma_1$ , which is also a minimum on  $\Gamma$ .

Obviously, similar considerations apply when  $J_0 = (-\infty, T'_0]$ ; and, indeed, when  $J_0$  and  $J_1$  both have these semi-infinite forms.

**9. Classical problems.** If  $U = E_m$  and the class of functions  $h_j$  is empty, then  $U(t, x) = E_m$  for all  $(t, x)$ , that is, there are no explicit restrictions (11). In this case, the fundamental condition (13) on  $u$  could come from a growth condition on  $f_0$  or some  $g_k$ , as discussed in § 7.

If we take  $f(t, x, u) = u$ , so that  $u = x'$  almost everywhere, then we have a minimum problem for  $(I) \int f_0(t, x, x') dt$ . The Tonelli theorem [16], on the existence of a minimum for nonparametric curve integrals, is just this problem with no explicit restrictions (11) and no integral restrictions (12); the condition (13) comes from a growth condition on  $f_0$ .

More generally, consider curves  $y: I \rightarrow E_l$  with absolutely continuous derivatives  $y^{(r-1)}$  of order  $r - 1$ . Take  $x = (x_{(1)}, x_{(2)}, \dots, x_{(r)})$

with  $x_{(\alpha)}(t) = y^{(\alpha-1)}(t)$ , and  $u(t) = y^{(r)}(t)$ . Then our work gives an existence theorem for the minimum of  $(I) \int f_0(t, y, y', \dots, y^{(r)}) dt$ . Here

$$f(t, x, u) = (x_{(2)}, x_{(3)}, \dots, x_{(r)}, u).$$

The linear bounding of the components of  $f$  is implied essentially by the bounding of  $(I) \int |y^{(r)}| dt$ .

Returning to first order problems, we can also consider parametric curve integrals  $(I) \int f_0(x, x') dt$  with  $f_0$  positively homogeneous of degree one in  $x'$ . In this case, a growth condition on  $f_0$  of the form previously considered is impossible. However, if there are no explicit restrictions (11), the functions  $g_k$  are similarly independent of  $t$  and positively homogeneous of degree one in  $x'$ ,  $G_0$  and  $G_1$  are constant, and  $B$  is of the form  $E_1 \times C$  for some closed set  $C$  in  $E_n$ , then we have a system invariant under Fréchet equivalence. We can reparametrize the curves of finite length  $L \neq 0$  by their relative arc lengths  $s/L$  on the interval  $I = [0, 1]$ ; here  $s$  is the arc length. In terms of the new parameter,  $|x'| = L$  almost everywhere. If the curves in  $\Gamma$  have bounded lengths, then  $\int |x'|$  is equi absolutely continuous. This is trivial for curves with  $L = 0$ . Thus condition (13) would be satisfied if the curves have bounded lengths.

We have really proved here part of Hilbert's theorem on compactness of a class of parametric curves. The bounding of the curve lengths  $L$  could come from the form of some  $g_k$

$$\left( \text{for example, } (I) \int g_k(x, x') dt \rightarrow \infty \text{ as } L \rightarrow \infty \right);$$

or, effectively, from the form of  $f_0$

$$\left( \text{for example, } (I) \int f_0(x, x') dt \rightarrow \infty \text{ as } L \rightarrow \infty \right).$$

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NAVAL ORDNANCE TEST STATION

Leonard Daniel Baumert, <i>Extreme copositive quadratic forms. II</i> .....	1
Edward Lee Bethel, <i>A note on continuous collections of disjoint continua</i> .....	21
Delmar L. Boyer and Adolf G. Mader, <i>A representation theorem for abelian groups with no elements of infinite p-height</i> .....	31
Jean-Claude B. Derderian, <i>Residuated mappings</i> .....	35
Burton I. Fein, <i>Representations of direct products of finite groups</i> .....	45
John Brady Garnett, <i>A topological characterization of Gleason parts</i> .....	59
Herbert Meyer Kamowitz, <i>On operators whose spectrum lies on a circle or a line</i> .....	65
Ignacy I. Kotlarski, <i>On characterizing the gamma and the normal distribution</i> .....	69
Yu-Lee Lee, <i>Topologies with the same class of homeomorphisms</i> .....	77
Moshe Mangad, <i>Asymptotic expansions of Fourier transforms and discrete polyharmonic Green's functions</i> .....	85
Jürg Thomas Marti, <i>On integro-differential equations in Banach spaces</i> ....	99
Walter Philipp, <i>Some metrical theorems in number theory</i> .....	109
Maxwell Alexander Rosenlicht, <i>Another proof of a theorem on rational cross sections</i> .....	129
Kenneth Allen Ross and Karl Robert Stromberg, <i>Jessen's theorem on Riemann sums for locally compact groups</i> .....	135
Stephen Simons, <i>A theorem on lattice ordered groups, results of Ptak, Namioka and Banach, and a front-ended proof of Lebesgue's theorem</i> .....	149
Morton Lincoln Slater, <i>On the equation <math>\varphi(x) = \int_x^{x+1} K(\xi) f[\varphi(\xi)] d\xi</math></i> .....	155
Arthur William John Stoddart, <i>Existence of optimal controls</i> .....	167
Burnett Roland Toskey, <i>A system of canonical forms for rings on a direct sum of two infinite cyclic groups</i> .....	179
Jerry Eugene Vaughan, <i>A modification of Morita's characterization of dimension</i> .....	189