SUB-STATIONARY PROCESSES

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This note supplements the longer paper [3]. It is proved in § 2 that if $T$ is a bounded Schwartz distribution on $\mathbb{R}^n$, e.g. an $L^\infty$ function, then its Fourier transform $\mathcal{F} T$ is of the form $\partial^n f / \partial t_1 \cdots \partial t_n$ where $f$ is integrable over any bounded set to any finite power. This follows from the main theorem of [3], but the proof here is much shorter.

Secondly, § 3 shows that a $p$-sub-stationary random (Schwartz) distribution has sample distributions of bounded order. This generalizes a result of K. Ito for the stationary case.

Third, in § 4 it is shown that $p$-sub-stationary stochastic processes define $p$-sub-stationary random distributions if $p \geq 1$.

In [5], K. Ito introduced stationary random Schwartz distributions $L$ with second moments. He obtained the “spectral measure” representation of the covariance of $L$. Using this, he proved for each such $L$:

(I) There is a finite $n$ such that almost all the sample distributions of $L$ are $n$th Schwartz derivatives of continuous functions.

The spectral measure also yields

(II) Almost all the sample distributions of $L$ are tempered distributions, and their Fourier transforms are first Schwartz derivatives of locally square-integrable functions.

In [3], (II) was proved for random distributions $L$ which are “$p$-sub-stationary” for some $p > 1$, i.e. for each $f$ in the Schwartz space $\mathcal{S}$,

$$
\sup_k E | L(\tau_k f) |^p < \infty ,
$$

where $(\tau_k f)(t) = f(t - k)$. Also, “locally square-integrable” was strengthened to “locally integrable to any finite power”. In § 2, we shall give corollaries of this result for fixed distributions and stochastic processes with much easier proofs. In § 3, we first prove (I) in the $p$-sub-stationary case for any $p > 0$, using some lemmas from [3] but no Fourier analysis. Then we obtain a result on the Fourier transform of the covariance for $p = 2$. In § 4, we show that for $p \geq 1$ a $p$-sub-stationary stochastic process is also a $p$-sub-stationary random distribution.
2. Fourier transforms of bounded functions and distributions.

All three theorems in this section are immediate corollaries of the main theorem of [3], but perhaps the easier proofs here will make that result more accessible.

We use the notations of L. Schwartz [8], e.g. $\mathcal{D}, \mathcal{D}', \mathcal{F}, \mathcal{F}'$. $\mathcal{F}$ is the Fourier transform operator. The results say that if a distribution $B$ is bounded or belongs to a suitable “stochastically bounded” class, then $\mathcal{F}B$ is of the following type:

**Definition.** A distribution $C$ in $\mathcal{D}'(R^k)$ is an $FB$-distribution ($C \in FB$) if and only if there is a measurable function $f$ on $R^k$ such that

$$C = \partial^k f/\partial t_1 \cdots \partial t_k$$

in the sense of distributions, and

$$\int_K |f(t)|^r dt_1 \cdots dt_k < \infty$$

whenever $0 < r < \infty$ and $K$ is compact.

Beurling [1] has called a distribution on $R$ a “pseudomeasure” if it is the first derivative of a locally integrable function. Thus the pseudomeasures include the class $FB$ on the real line. The work of Beurling, Kahane and Salem [6] and others on pseudomeasures has apparently been primarily devoted to the question of which compact sets carry pseudomeasures of certain types. I do not know of any mutual implications between our results.

A distribution $B$ in $\mathcal{D}'(R^k)$ is called bounded ($B \in \mathcal{B}'$) if for every $f$ in $\mathcal{D}$,

$$\sup \{|B(\tau_h f)| : h \in R^k\} < \infty$$

(cf. Schwartz [8, tome I, Théorème IX(b) p. 72; tome II, “Autre définition des distributions bornées”, p. 61]). It follows immediately from the main theorem of [3] that if $B \in \mathcal{B}'$, then $\mathcal{F}B \in FB$.

We shall use here the Hausdorff-Young inequality for Fourier transforms rather than for series as in [3]. Suppose $1 < p \leq 2$, $q = p/(p - 1)$, and $f \in L^p(R)$. Let

$$f_n(t) = \begin{cases} f(t), & |t| \leq n \\ 0, & |t| > n \end{cases}$$

Then the functions $\mathcal{F}f_n$ are in $L^q(R)$, and for some $h$ in $L^q(R)$, $\mathcal{F}f_n \to h$ in $L^q$ (Zygmund [9, 12.41 p. 316]). In the sense of tempered distributions, we have simply $\mathcal{F}f = h$. 
To illustrate our method, we first prove

**Theorem 2.1.** If \( f \in L^p(R) \), then \( \mathcal{F}f \in FB \).

*Proof.* Let \( g(t) = f(t) \) for \( |t| \leq 1 \), \( g(t) = 0 \) elsewhere, and \( h = f - g \). Then by the Paley-Wiener theorem, \( \mathcal{F}g \) is an entire analytic function, hence so is its indefinite integral, and \( \mathcal{F}g \in FB \).

Let \( j(t) = h(t)/t \). Then \( j \in L^p(R) \) for all \( p > 1 \), so \( \mathcal{F}j \in L^s \) for all \( q \geq 2 \). Thus

\[
D(\mathcal{F}j) = \mathcal{F}(-2\pi itj) = \mathcal{F}(-2\pi ih) \in FB,
\]

so \( \mathcal{F}h \in FB \). Hence \( \mathcal{F}f \in FB \).

In [3], there was an example of a bounded function \( f \) (the Heaviside function) with \( \mathcal{F}f = D\phi \), so that \( \phi \in L^r \) on each bounded set for \( r \) finite, but with \( \phi \) unbounded near zero.

Next suppose \((\Omega, \mathcal{B}, P)\) is a probability space. A jointly measurable map

\[
\langle t, \omega \rangle \rightarrow x(t, \omega)
\]

of \( R^k \times \Omega \) into \( R \) will be called a *measurable stochastic process* on \( R^k \), which is *p-sub-stationary* if

\[
\sup_t \int |x(t, \omega)|^p \, dP(\omega) = M < \infty.
\]

We let \( X_\omega(t) = x(t, \omega) \), and \( E = \text{integral with respect to } P \).

**Theorem 2.2.** Suppose \( x(\cdot, \cdot) \) is a p-sub-stationary process on \( R \) and \( p > 1 \). Then for \( P \)-almost all \( \omega \), \( \mathcal{F}X_\omega \in FB \).

*Proof.* Let \( Y_\omega(t) = X_\omega(t) \) for \( |t| \leq 1 \), \( Y_\omega(t) = 0 \) elsewhere, and \( Z_\omega = X_\omega - Y_\omega \). Then for \( 1 < r \leq p \),

\[
E \int_{-\infty}^{\infty} |Z_\omega(t)|^r \, dt \leq \int_{|t| \geq 1} (E |X_\omega(t)|^p)^{r/p} |t|^r \, dt \leq 2M^r(r - 1).
\]

Thus \( Z_\omega(t)/t \in L^r \) for almost all \( \omega \), so

\[
\mathcal{F}(Z_\omega(t)/t) \in L^s \quad \text{for } \frac{p}{p-1} \leq s < \infty.
\]

Thus \( D\mathcal{F}(Z_\omega(t)/t) \in FB \), and hence \( \mathcal{F}Z_\omega \in FB \). Now \( Y_\omega \) is almost surely integrable with compact support, so \( \mathcal{F}Y_\omega \) and its indefinite integral are entire functions, \( \mathcal{F}Y_\omega \in FB \), and \( \mathcal{F}X_\omega \in FB \) for almost all \( \omega \).

Now we generalize Theorem 2.1:

**Theorem 2.3.** If \( T \in \mathcal{B}'(R^k) \), then \( \mathcal{F}T \in FB \).
Proof. $T$ is a finite sum of partial derivatives of bounded functions (Schwartz [8, tome II, Théorème XXV p. 57]). Clearly $FB$ is closed under multiplication by polynomials. Thus we may assume $T$ is a function $f$ in $L^\infty(R^k)$.

For each subset $A$ of the finite set $\{1, 2, \ldots, k\}$, let $S_A$ be the set of all $t$ in $R^k$ such that $|t_j| > 1$ if and only if $j \in A$. Let $f_A = f$ on $S_A, f_A = 0$ elsewhere. Then for each $A$,

$$
g_A = f_A/\prod_{j \in A} t_j \in L^p(R^k) \quad \text{for all } p > 1,
$$

so that $\mathcal{F}g_A \in L^q(R^k)$ for all $q \geq 2$. Taking indefinite integrals in the $x_j$ for $j \in A$, we obtain $\mathcal{F}f_A = \partial^k h_A/\partial x_1 \cdots \partial x_k$, where

$$
\int_K |h_A(x)|^r \, dx_1 \cdots dx_k < \infty
$$

whenever $0 < r < \infty$ and $K$ is compact. Thus

$$\mathcal{F}f = \sum_A \mathcal{F}f_A \in FB.$$  

The converse of Theorem 2.3 is not true, since it is easy to construct examples of 2-sub-stationary stochastic processes whose sample functions are unbounded (as distributions) with probability 1.

3. $p$-sub-stationary random distributions are of finite order. Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $M(\Omega)$ be the linear space of $\mathcal{B}$-measurable complex-valued functions on $\Omega$ modulo functions which vanish $P$-almost everywhere. On $M(\Omega)$, let $T(P)$ be the topology of convergence in probability. $T(P)$ is metrizable, e.g. by the metric

$$
d(f, g) = \int \|f(x) - g(x)/(1 + |f(x) - g(x)|)\|dP(x),
$$

but it is not locally convex in general.

DEFINITION. A random distribution is a sequentially continuous linear map from $\mathcal{D}(R)$ into some $M(\Omega)$ with topology $T(P)$.

It follows from a theorem of R. A. Minlos [7] (see [4, Chapter 4, § 2, # 4, Theorem 6]) that for any random distribution $L$ there is a countably additive measure $Q$ on $\mathcal{D}'$ such that for any $f_1, \cdots, f_n$ in $\mathcal{D}$ and Borel set $B \subset C^n$,

$$Q(M: \langle M(f_1), \cdots, M(f_n) \rangle \in B) = P(\omega: \langle L(f_1)(\omega), \cdots, L(f_n)(\omega) \rangle \in B).$$

The subsets of $\mathcal{D}'$ on which $Q$ is given form an algebra (the "cylinder sets"). The unique countably additive extension of $Q$ to the
generated \(\sigma\)-algebra will be called the \textit{Minlos measure} of \(L\).

For any \(f\) in \(\mathcal{D}(\mathbb{R})\) and integer \(n \geq 0\) we let

\[
\|f\|_n = \left( \sum_{j=0}^{n} \int_{-\infty}^{\infty} |D^j f(x)|^p \, dx \right)^{1/p}.
\]

Also, for any finite interval \((a, b)\), \(\mathcal{D}[a, b]\) will denote the space of \(C^\infty\) functions vanishing outside \((a, b)\), with its relative topology from \(\mathcal{D}\). This relative topology is defined by the countably many norms \(\| \|_n\) (although that of \(\mathcal{D}\) is not). For \(A\) and \(B\) in \(\mathcal{D}'\) we say \(A = B\) on \((a, b)\)" if \(A(f) = B(f)\) for all \(f\) in \(\mathcal{D}[a, b]\). The distribution defined by a locally integrable function \(f\) or derivative \(D^p f\) will be written \([f]\) or \([D^p f]\) respectively.

Clearly a continuous linear functional \(A\) on \(\mathcal{D}[a, b]\) for \(\| \|_n\) has the form

\[
A(f) = \sum_{j=0}^{n} \int_{a}^{b} D^j f(x) \overline{g}_j(x) \, dx
\]

for some \(g_j\) in \(L^2[a, b]\). Thus, integrating by parts and adding, we have

\[
A(f) = [D^ng](f) = [D^{n+1}h](f)
\]

for some \(g\) in \(L^2(a, b)\) and absolutely continuous \(h\) on \((a, b)\).

**Theorem 3.1.** Let \(L\) be a \(p\)-sub-stationary random distribution for some \(p > 0\). Then there is a positive integer \(n\) such that the Minlos measure of \(L\) is concentrated in the set of \(M\) in \(\mathcal{D}'\) such that \(M = D^n f\) for some continuous function \(f\) (depending on \(M\)).

**Proof.** The hypothesis becomes stronger as \(p\) increases. Thus we may assume \(0 < p \leq 1\). For each \(g\) in \(\mathcal{D}\) let

\[
A(g) = \sup_{\tau} (E \mid L(\tau, g) \mid^p)^{1/p} < \infty.
\]

Note that \(A\) will not generally be a pseudo-norm for \(p < 1\). By Lemma 4 of [3], there exist \(K\) and \(n \geq 0\) such that \(A(g) \leq K \|g\|_n\) for all \(g\) in \(\mathcal{D}[0, 1]\), hence for \(g\) in \(\mathcal{D}[b, b+1]\) for any real \(b\).

Now given \(c > 0\), there exist \(f_1, \cdots, f_m\) in \(\mathcal{D}\) such that

\[
\sum_{j=1}^{m} f_j(t) = 1 \quad \text{for } |t| \leq c,
\]

and such that the diameter of the support of each \(f_j\) is at most 1 (cf. [3, proof of Lemma 5]). Let \(g \in \mathcal{D}[-c, c]\). Then for each \(j\),
Thus for some $M_c > 0$,
\[
A(g) = \left( \left( \sum_{j=1}^{m} (gf_j) \right)^{p} \right)^{1/p} \leq \left( \sum_{j=1}^{m} (A(gf_j))^{p} \right)^{1/p} \leq K \left( \sum_{j=1}^{m} \|gf_j\|_\infty \right)^{1/p} \leq M_c \|g\|_\infty
\]
for all $g$ in $\mathcal{D}[-c, c]$.

Now $\mathcal{D}[-c, c]$ is a nuclear space (see e.g. Gelfand and Vilenkin [4, Chapter I, §3, #6]). Thus a theorem essentially due to Minlos ([7], [4, Chapter IV, §2, #3, Theorem 4]) implies that the Minlos measure of $L$ restricted to $\mathcal{D}[-c, c]$ is concentrated in the set of distributions continuous for $\|f\|_{n+r}$ for some $r$ (actually $r = 1$). Thus the Minlos measure is concentrated in the set of all $M$ of the form

$$M = [D^{n+r+1}f]$$

where $f$ is continuous and depends on $M$. Given $M$, $f$ on $(-c, c)$ is determined up to an additive polynomial of degree at most $n + r$. Fixing $f$ on $(-1, 1)$, say, we obtain

$$M = [D^{n+r+1}f]$$

for some continuous $f$ (not necessarily bounded on $R$). The proof is complete.

A simpler form of the last proof yields

**Theorem 3.2.** Let $L$ be a random distribution, $p > 0$, and $(a, b)$ a finite interval. Suppose $E|L(f)|^p < \infty$ for all $f$ in $\mathcal{D}[a, b]$. Then for some $n$, the Minlos measure of $L$ is concentrated in the set of all $M$ in $\mathcal{D}'$ equal on $(a, b)$ to $[D^n f]$ for $f$ continuous on $[a, b]$.

**Proof.** $L$ is continuous from $\mathcal{D}[a, b]$ to $L^p(\Omega)$ [3, Lemma 2]. Thus for some $n$ and $\varepsilon > 0$,

$$\|f\|_n < \varepsilon \quad \text{implies} \quad E|L(f)|^p < 1,$$

and

$$(E|L(f)|^p)^{1/p} \leq \|f\|_n/\varepsilon$$
for all \( f \) in \( \mathcal{D}[a, b] \) by homogeneity. Now we use nuclearity of \( \mathcal{D}[a, b] \) and can proceed as in the last proof.

Suppose \( L \) is a random distribution with finite second moments, i.e., its range in \( M(\Omega) \) is included in \( L^2(\Omega, \mathcal{A}, P) \). Then there is a unique \( B \) in \( \mathcal{D}'(R^2) \) such that

\[
E(L(f)L(g)) = B(f \otimes g)
\]

where \( (f \otimes g)(s, t) = f(s)g(t) \) (see e.g. [2, §3]).

**Lemma.** If \( L \) is 2-sub-stationary, then \( B \) is bounded.

**Proof.** We must show that for any \( h \) in \( \mathcal{D}(R^2) \), \( B(\tau_z h) \) remains bounded as \( z \) runs over \( R^2 \). We know this for \( h \) of the form \( f \otimes g \), \( f, g \in \mathcal{D}(R) \).

For a general \( h \), we have \( h(s, t) = 0 \) outside some square \( C_M: |s| \leq M, |t| \leq M \). Let \( g \in \mathcal{D}(R) \), \( g(s) = 1 \) for \( |s| \leq M \), and \( g(s) = 0 \) for \( |s| \geq 2M \). We expand \( h \) in a Fourier series:

\[
h(s, t) = g(s)g(t) \sum_{m, n} a(m, n) \exp(i(\pi ms + nt)/2M)
\]

for all \((s, t)\) in \( R^2 \). Since \( h \) on \( C_M \) extends to a \( C^\infty \) function periodic of period \( 4M \) in \( s \) and \( t \), we know that for any polynomial \( p \) in two variables, \( p(m, n)a(m, n) \) is bounded.

Now, by Lemma 4 of [3] there exist \( k \) and \( N > 0 \) such that

\[
\sup_{\omega} (E | L(\tau_{\omega} f) |^2)^{1/2} \leq N \| f \|_h
\]

for all \( f \) in \( \mathcal{D}[-2M, 2M] \). Let

\[
h_m(s) = g(s) \exp(i ms / 2M).
\]

Then

\[
\| h_m \|_k = \left( \sum_{j=0}^{k} \int_{-2M}^{2M} | D^j h_m(s) |^2 ds \right)^{1/2} \leq T(1 + m^5)_k
\]

for some \( T > 0 \) (depending on \( M \) and \( g \), but not on \( m \)). Now

\[
h(s, t) = \sum_{m, n} a(m, n) h_m(s) h_n(t)
\]

and \( a(m, n)(1 + m^5)^{k+1}(1 + n^5)^{k+1} \) is bounded in \( m \) and \( n \), so

\[
\sup_{z} | B(\tau_z h) | \leq \sup_{z, t} \sum_{m, n} | a(m, n) B(\tau_z h_m \otimes \tau_t h_n) |
\]

\[
\leq N^2 \sum_{m, n} | a(m, n) | \| h_m \|_k \| h_n \|_k < \infty.
\]

From the lemma just proved and Theorem 2.3, we can infer that
for any 2-sub-stationary random distribution $L$,

$$E(L(f)L(g)) = C(\mathcal{F}f \otimes (\mathcal{F}g)^\vee)$$

for some $\mathcal{F}B$-distribution $C$, i.e.

$$C = [\partial^2 f(x, y)/\partial x \partial y]$$

for some measurable function $f$ integrable to any finite power over any compact set. When $f$ is of bounded variation on $\mathbb{R}^h$, $L$ (or $B$) is called harmonizable. Clearly such a $B$ is a bounded continuous function: $B \in \mathcal{C}(\mathbb{R}^h)$. We have the following inclusions of subsets of $\mathcal{D}'(\mathbb{R}^h)$:

$$\text{harmonizable} \subset \mathcal{C} \subset L^\infty \subset \mathcal{B}' \subset \mathcal{F}^{-1}(\mathcal{F}B) \subset \mathcal{F}^{-1} \text{ (pseudomeasures)}.$$

For none of these classes do we have a simple characterization both of the distributions and of their Fourier transforms (such as the Bochner, Plancherel and Paley-Wiener theorems and their generalizations and other results of Schwartz). Thus which will yield the most useful theory remains unclear.

4. Stochastic processes and random distributions.

**Theorem 4.1.** If $p \geq 1$, a $p$-sub-stationary stochastic process $x(\cdot, \cdot)$ is a $p$-sub-stationary random distribution.

**Proof.** Let $f \in \mathcal{D}(\mathbb{R}^h)$. For any $h$ in $\mathbb{R}^h$, let

$$A(f, h) = \left\{ \left[ \int_{\mathbb{R}^h} f(t - h)x(t, \omega)dt \right]^p dP(\omega) \right\} \geq \left[ \int_{\mathbb{R}^h} f(s)x(s + h, \omega)ds \right]^p dP(\omega).$$

Let $C$ be the support of $f$ and let $\lambda$ be Lebesgue measure. We apply Hölder's inequality to the inner integral, with $q = p/(p - 1)$, obtaining

$$A(f, h) \leq ||f||_p^p \int_{\omega} |x(s, \omega)|^p ds dP(\omega) \leq ||f||_p^p \lambda(C) \sup_s \int |x(s, \omega)|^p dP(\omega) < \infty.$$

Thus a random distribution $L$ is defined by

$$L(f)(\omega) = \int_{\mathbb{R}^h} x(t, \omega)f(t)dt.$$
and is $p$-sub-stationary.

For $p < 1$, it seems unclear whether a $p$-sub-stationary stochastic process defines a random distribution at all.

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