SUB-STATIONARY PROCESSES

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This note supplements the longer paper [3]. It is proved in § 2 that if $T$ is a bounded Schwartz distribution on $\mathbb{R}^n$, e.g. an $L^\infty$ function, then its Fourier transform $\widehat{T}$ is of the form $\partial^a f/\partial t_1 \cdots \partial t_n$ where $f$ is integrable over any bounded set to any finite power. This follows from the main theorem of [3], but the proof here is much shorter.

Secondly, § 3 shows that a $p$-sub-stationary random (Schwartz) distribution has sample distributions of bounded order. This generalizes a result of K. Ito for the stationary case.

Third, in § 4 it is shown that $p$-sub-stationary stochastic processes define $p$-sub-stationary random distributions if $p \geq 1$.

In [5], K. Ito introduced stationary random Schwartz distributions $L$ with second moments. He obtained the “spectral measure” representation of the covariance of $L$. Using this, he proved for each such $L$:

(I) There is a finite $n$ such that almost all the sample distributions of $L$ are $n$th Schwartz derivatives of continuous functions.

The spectral measure also yields

(II) Almost all the sample distributions of $L$ are tempered distributions, and their Fourier transforms are first Schwartz derivatives of locally square-integrable functions.

In [3], (II) was proved for random distributions $L$ which are “$p$-sub-stationary” for some $p > 1$, i.e. for each $f$ in the Schwartz space $\mathcal{S}$,

$$\sup_h E |L(\tau_h f)|^p < \infty,$$

where $(\tau_h f)(t) = f(t - h)$. Also, “locally square-integrable” was strengthened to “locally integrable to any finite power”. In § 2, we shall give corollaries of this result for fixed distributions and stochastic processes with much easier proofs. In § 3, we first prove (I) in the $p$-sub-stationary case for any $p > 0$, using some lemmas from [3] but no Fourier analysis. Then we obtain a result on the Fourier transform of the covariance for $p = 2$. In § 4, we show that for $p \geq 1$ a $p$-sub-stationary stochastic process is also a $p$-sub-stationary random distribution.
2. Fourier transforms of bounded functions and distributions. All three theorems in this section are immediate corollaries of the main theorem of [3], but perhaps the easier proofs here will make that result more accessible.

We use the notations of L. Schwartz [8], e.g. \( \mathcal{D}, \mathcal{D}', \mathcal{S}, \mathcal{S}' \). \( \mathcal{F} \) is the Fourier transform operator. The results say that if a distribution \( B \) is bounded or belongs to a suitable "stochastically bounded" class, then \( \mathcal{F}B \) is of the following type:

**DEFINITION.** A distribution \( C \) in \( \mathcal{D}'(R^k) \) is an \textit{FB-distribution} \((C \in FB)\) if and only if there is a measurable function \( f \) on \( R^k \) such that

\[
C = \frac{\partial^k f}{\partial t_1 \cdots \partial t_k}
\]

in the sense of distributions, and

\[
\int_K |f(t)|^r dt_1 \cdots dt_k < \infty
\]

whenever \( 0 < r < \infty \) and \( K \) is compact.

Beurling [1] has called a distribution on \( R \) a "pseudomeasure" if it is the first derivative of a locally integrable function. Thus the pseudomeasures include the class \( FB \) on the real line. The work of Beurling, Kahane and Salem [6] and others on pseudomeasures has apparently been primarily devoted to the question of which compact sets carry pseudomeasures of certain types. I do not know of any mutual implications between our results.

A distribution \( B \) in \( \mathcal{D}'(R^k) \) is called \textit{bounded} \((B \in \mathcal{B}')\) if for every \( f \) in \( \mathcal{D} \),

\[
\sup \{|B(\tau_h f)| : h \in R^k\} < \infty
\]

(cf. Schwartz [8, tome I, Théorème IX(b) p. 72; tome II, "Autre définition des distributions bornées", p. 61]). It follows immediately from the main theorem of [3] that if \( B \in \mathcal{B}' \), then \( \mathcal{F}B \in FB \).

We shall use here the Hausdorff-Young inequality for Fourier transforms rather than for series as in [3]. Suppose \( 1 < p \leq 2, q = p/(p - 1) \), and \( f \in L^q(R) \). Let

\[
f_n(t) = \begin{cases} f(t), & |t| \leq n \\ 0, & |t| > n \end{cases}.
\]

Then the functions \( \mathcal{F}f_n \) are in \( L^q(R) \), and for some \( h \) in \( L^q(R) \), \( \mathcal{F}f_n \to h \) in \( L^q \) (Zygmund [9, 12.41 p. 316]). In the sense of tempered distributions, we have simply \( \mathcal{F}f = h \).
To illustrate our method, we first prove

**Theorem 2.1.** If \( f \in L^\infty(\mathbb{R}) \), then \( \mathcal{F} f \in FB \).

*Proof.* Let \( g(t) = f(t) \) for \( |t| \leq 1 \), \( g(t) = 0 \) elsewhere, and \( h = f - g \). Then by the Paley-Wiener theorem, \( \mathcal{F} g \) is an entire analytic function, hence so is its indefinite integral, and \( \mathcal{F} g \in FB \).

Let \( j(t) = h(t)/t \). Then \( j \in L^p(\mathbb{R}) \) for all \( p > 1 \), so \( \mathcal{F} j \in L^q \) for all \( q \geq 2 \). Thus

\[
D(\mathcal{F} j) = \mathcal{F}(-2\pi i t j) = \mathcal{F}(-2\pi i h) \in FB,
\]

so \( \mathcal{F} h \in FB \). Hence \( \mathcal{F} f \in FB \).

In [3], there was an example of a bounded function \( f \) (the Heaviside function) with \( \mathcal{F} f = D\phi \), so that \( \phi \in L^r \) on each bounded set for \( r \) finite, but with \( \phi \) unbounded near zero.

Next suppose \((\Omega, \mathcal{B}, P)\) is a probability space. A jointly measurable map

\[
\langle t, \omega \rangle \rightarrow x(t, \omega)
\]

of \( \mathbb{R}^k \times \Omega \) into \( \mathbb{R} \) will be called a *measurable stochastic process* on \( \mathbb{R}^k \), which is \( p \)-sub-stationary if

\[
\sup_t \left| x(t, \omega) \right|^p dP(\omega) = M < \infty.
\]

We let \( X_\omega(t) = x(t, \omega) \), and \( E = \text{integral with respect to } P \).

**Theorem 2.2.** Suppose \( x(\cdot, \cdot) \) is a \( p \)-sub-stationary process on \( \mathbb{R} \) and \( p > 1 \). Then for \( P \)-almost all \( \omega \), \( \mathcal{F} X_\omega \in FB \).

*Proof.* Let \( Y_\omega(t) = X_\omega(t) \) for \( |t| \leq 1 \), \( Y_\omega(t) = 0 \) elsewhere, and \( Z_\omega = X_\omega - Y_\omega \). Then for \( 1 < r \leq p \),

\[
E \int_{-\infty}^{\infty} \left| Z_\omega(t)/t \right|^r dt \leq \int_{|t| \geq 1} (E \left| X_\omega(t) \right|^p)^{r/p} \left| t \right|^r dt \leq 2M^r/(r - 1) .
\]

Thus \( Z_\omega(t)/t \in L^r \) for almost all \( \omega \), so

\[
\mathcal{F} (Z_\omega(t)/t) \in L^s \quad \text{for } s \geq p/(p - 1) \leq s < \infty.
\]

Thus \( D\mathcal{F} (Z_\omega(t)/t) \in FB \), and hence \( \mathcal{F} Z_\omega \in FB \). Now \( Y_\omega \) is almost surely integrable with compact support, so \( \mathcal{F} Y_\omega \) and its indefinite integral are entire functions, \( \mathcal{F} Y_\omega \in FB \), and \( \mathcal{F} X_\omega \in FB \) for almost all \( \omega \).

Now we generalize Theorem 2.1:

**Theorem 2.3.** If \( T \in \mathcal{B}'(\mathbb{R}^k) \), then \( \mathcal{F} T \in FB \).
Proof. \( T \) is a finite sum of partial derivatives of bounded functions (Schwartz [8, tome II, Théorème XXV p. 57]). Clearly \( FB \) is closed under multiplication by polynomials. Thus we may assume \( T \) is a function \( f \) in \( L^\infty(R^k) \).

For each subset \( A \) of the finite set \( \{1, 2, \ldots, k\} \), let \( S_A \) be the set of all \( t \) in \( R^k \) such that \( |t_j| > 1 \) if and only if \( j \in A \). Let \( f_A = f \) on \( S_A \), \( f_A = 0 \) elsewhere. Then for each \( A \),

\[
g_A = f_A/\prod_{j \in A} t_j \in L^p(R^k) \quad \text{for all } p > 1,
\]

so that \( \mathcal{F} g_A \in L^q(R^k) \) for all \( q \geq 2 \). Taking indefinite integrals in the \( x_j \) for \( j \in A \), we obtain \( \mathcal{F} f_A = \partial^\beta h_A/\partial x_1 \cdots \partial x_k \), where

\[
\int_K |h_A(x)|^r \, dx_1 \cdots dx_k < \infty
\]

whenever \( 0 < r < \infty \) and \( K \) is compact. Thus

\[
\mathcal{F} f = \sum_A \mathcal{F} f_A \in FB.
\]

The converse of Theorem 2.3 is not true, since it is easy to construct examples of 2-sub-stationary stochastic processes whose sample functions are unbounded (as distributions) with probability 1.

3. \( p \)-sub-stationary random distributions are of finite order. Let \( (\Omega, \mathcal{B}, P) \) be a probability space and let \( M(\Omega) \) be the linear space of \( \mathcal{B} \)-measurable complex-valued functions on \( \Omega \) modulo functions which vanish \( P \)-almost everywhere. On \( M(\Omega) \), let \( T(P) \) be the topology of convergence in probability. \( T(P) \) is metrizable, e.g. by the metric

\[
d(f, g) = \int |f(x) - g(x)|/(1 + |f(x) - g(x)|) \, dP(x),
\]

but it is not locally convex in general.

DEFINITION. A random distribution is a sequentially continuous linear map from \( \mathcal{D}(R) \) into some \( M(\Omega) \) with topology \( T(P) \).

It follows from a theorem of R. A. Minlos [7] (see [4, Chapter 4, § 2, # 4, Theorem 6]) that for any random distribution \( L \) there is a countably additive measure \( Q \) on \( \mathcal{D}' \) such that for any \( f_1, \cdots, f_n \) in \( \mathcal{D} \) and Borel set \( B \subset C^* \),

\[
Q\{M: \langle M(f_1), \cdots, M(f_n) \rangle \in B\} = P\{\omega: \langle L(f_1)(\omega), \cdots, L(f_n)(\omega) \rangle \in B\}.
\]

The subsets of \( \mathcal{D}' \) on which \( Q \) is given form an algebra (the “cylinder sets”). The unique countably additive extension of \( Q \) to the
generated $\sigma$-algebra will be called the Minlos measure of $L$.

For any $f$ in $\mathcal{D}(R)$ and integer $n \geq 0$ we let

$$||f||_n = \left(\sum_{j=0}^{n} \int_{-\infty}^{\infty} |D^j f(x)|^2 dx \right)^{1/2}.$$ 

Also, for any finite interval $(a, b)$, $\mathcal{D}[a, b]$ will denote the space of $C^\infty$ functions vanishing outside $(a, b)$, with its relative topology from $\mathcal{D}$. This relative topology is defined by the countably many norms $||\cdot||_n$ (although that of $\mathcal{D}$ is not). For $A$ and $B$ in $\mathcal{D}'$ we say “$A = B$ on $(a, b)$” if $A(f) = B(f)$ for all $f$ in $\mathcal{D}[a, b]$. The distribution defined by a locally integrable function $f$ or derivative $D^p f$ will be written $[f]$ or $[D^p f]$ respectively.

Clearly a continuous linear functional $A$ on $\mathcal{D}[a, b]$ for $||\cdot||_n$ has the form

$$A(f) = \sum_{j=0}^{n} \int_{a}^{b} D^j f(x) \tilde{g}_j(x) dx$$

for some $g_j$ in $L^2[a, b]$. Thus, integrating by parts and adding, we have

$$A(f) = [D^n g](f) = [D^{n+1} h](f)$$

for some $g$ in $L^2(a, b)$ and absolutely continuous $h$ on $(a, b)$.

**Theorem 3.1.** Let $L$ be a $p$-sub-stationary random distribution for some $p > 0$. Then there is a positive integer $n$ such that the Minlos measure of $L$ is concentrated in the set of $M$ in $\mathcal{D}'$ such that $M = D^n f$ for some continuous function $f$ (depending on $M$).

**Proof.** The hypothesis becomes stronger as $p$ increases. Thus we may assume $0 < p \leq 1$. For each $g$ in $\mathcal{D}$ let

$$A(g) = \sup_t (E|L(\tau_t g)|^p)^{1/p} < \infty.$$ 

Note that $A$ will not generally be a pseudo-norm for $p < 1$. By Lemma 4 of [3], there exist $K$ and $n \geq 0$ such that $A(g) \leq K ||g||_n$ for all $g$ in $\mathcal{D}[0, 1]$, hence for $g$ in $\mathcal{D}[b, b+1]$ for any real $b$.

Now given $c > 0$, there exist $f_1, \cdots, f_m$ in $\mathcal{D}$ such that

$$\sum_{j=1}^{m} f_j(t) = 1 \text{ for } |t| \leq c,$$

and such that the diameter of the support of each $f_j$ is at most 1 (cf. [3, proof of Lemma 5]). Let $g \in \mathcal{D}[-c, c]$. Then for each $j$,
\[ \| gf_j \|_n = \left( \sum_{p=0}^\infty \left| D^p(gf_j) \right|^2 dt \right)^{1/2} \]
\[ = \left( \sum_{p=0}^\infty \left( \sum_{q=0}^p \binom{p}{q} D^q g D^{p-q}f_j \right)^2 dt \right)^{1/2} \]
\[ \leq (n + 1)2^n \| g \|_n \max (|D^r f_j(t)| : t \in \mathbb{R}, 0 \leq r \leq n) . \]

Thus for some \( M_c > 0, \)
\[ A(g) = \left( \left( \sum_{j=1}^m (gf_j) \right)^p \right)^{1/p} \leq \left( \sum_{j=1}^m (A(gf_j))^p \right)^{1/p} \]
\[ \leq K \left( \sum_{j=1}^m \| gf_j \|_n^p \right)^{1/p} \leq M_c \| g \|_n \]
for all \( g \) in \( \mathcal{D}[-c, c] . \)

Now \( \mathcal{D}[-c, c] \) is a nuclear space (see e.g. Gelfand and Vilenkin [4, Chapter I, §3, #6]). Thus a theorem essentially due to Minlos ([7], [4, Chapter IV, §2, #3, Theorem 4]) implies that the Minlos measure of \( L \) restricted to \( \mathcal{D}[-c, c] \) is concentrated in the set of distributions continuous for \( \| \|_{n+r} \) for some \( r \) (actually \( r = 1 \)). Thus the Minlos measure is concentrated in the set of all \( M \) of the form
\[ M = [D^{n+r+1}f] \] on \((-c, c)\)
where \( f \) is continuous and depends on \( M \). Given \( M, f \) on \((-c, c)\) is determined up to an additive polynomial of degree at most \( n + r \). Fixing \( f \) on \((-1, 1)\), say, we obtain
\[ M = [D^{n+r+1}f] \]
for some continuous \( f \) (not necessarily bounded on \( \mathbb{R} \)). The proof is complete.

A simpler form of the last proof yields

**Theorem 3.2.** Let \( L \) be a random distribution, \( p > 0 \), and \((a, b)\) a finite interval. Suppose \( E|L(f)|^p < \infty \) for all \( f \) in \( \mathcal{D}[a, b] . \) Then for some \( n \), the Minlos measure of \( L \) is concentrated in the set of all \( M \) in \( \mathcal{D}^\ast \) equal on \((a, b)\) to \([D^n f] \) for \( f \) continuous on \([a, b] \).

**Proof.** \( L \) is continuous from \( \mathcal{D}[a, b] \) to \( L^p(\Omega) \) [3, Lemma 2]. Thus for some \( n \) and \( \varepsilon > 0, \)
\[ \| f \|_n < \varepsilon \quad \text{implies} \quad E|L(f)|^p < 1 , \]
and
\[ (E|L(f)|^p)^{1/p} \leq \| f \|_n/\varepsilon \]
for all \( f \) in \( \mathcal{D}[a, b] \) by homogeneity. Now we use nuclearity of \( \mathcal{D}[a, b] \) and can proceed as in the last proof.

Suppose \( L \) is a random distribution with finite second moments, i.e., its range in \( M(\Omega) \) is included in \( L^2(\Omega, \mathcal{B}, P) \). Then there is a unique \( B \) in \( \mathcal{D}'(\mathbb{R}^2) \) such that

\[
E(L(f)\overline{L(g)}) = B(f \otimes \overline{g})
\]

where \( (f \otimes \overline{g})(s, t) = f(s)\overline{g}(t) \) (see e.g. [2, §3]).

**Lemma.** If \( L \) is 2-sub-stationary, then \( B \) is bounded.

**Proof.** We must show that for any \( h \) in \( \mathcal{D}(\mathbb{R}^2) \), \( B(\tau_z h) \) remains bounded as \( z \) runs over \( \mathbb{R}^2 \). We know this for \( h \) of the form \( f \otimes g \), \( f, g \in \mathcal{D}(\mathbb{R}) \).

For a general \( h \), we have \( h(s, t) = 0 \) outside some square \( C_M: |s| \leq M, |t| \leq M \). Let \( g \in \mathcal{D}(\mathbb{R}), g(s) = 1 \) for \( |s| \leq M \), and \( g(s) = 0 \) for \( |s| \geq 2M \). We expand \( h \) in a Fourier series:

\[
h(s, t) = g(s)g(t) \sum_{m, n} a(m, n) \exp(\pi i(ms + nt)/2M)
\]

for all \( (s, t) \) in \( \mathbb{R}^2 \). Since \( h \) on \( C_{2M} \) extends to a \( C^\infty \) function periodic of period \( 4M \) in \( s \) and \( t \), we know that for any polynomial \( p \) in two variables, \( p(m, n)a(m, n) \) is bounded.

Now, by Lemma 4 of [3] there exist \( k \) and \( N > 0 \) such that

\[
\sup_u (E|L(\tau_u f)|^{1/2}) \leq N\|f\|_k
\]

for all \( f \) in \( \mathcal{D}[-2M, 2M] \). Let

\[
h_m(s) = g(s) \exp(\pi ims/2M) .
\]

Then

\[
\|h_m\|_k = \left(\sum_{j=0}^k \int_{-M}^M |D^j h_m(s)|^2 ds\right)^{1/2} \leq T(1 + m^2)^k
\]

for some \( T > 0 \) (depending on \( M \) and \( g \), but not on \( m \)). Now

\[
h(s, t) = \sum_{m, n} a(m, n)h_m(s)h_n(t)
\]

and \( a(m, n)(1 + m^2)^{k+1}(1 + n^2)^{k+1} \) is bounded in \( m \) and \( n \), so

\[
\sup_x |B(\tau_x h)| \leq \sup_{s, f} \sum_{m, n} |a(m, n)| \|B(\tau_s h_m \otimes \tau_t h_n)\| \leq N^2 \sum_{m, n} |a(m, n)| \|h_m\| \|h_n\| < \infty .
\]

From the lemma just proved and Theorem 2.3, we can infer that
for any 2-sub-stationary random distribution $L$,

$$E(L(f)L(g)) = C(\mathcal{F} f \otimes (\mathcal{F} g)^-)$$

for some $FB$-distribution $C$, i.e.

$$C = \left[ \partial^2 f(x, y) / \partial x \partial y \right]$$

for some measurable function $f$ integrable to any finite power over any compact set. When $f$ is of bounded variation on $R^2$, $L$ (or $B$) is called harmonizable. Clearly such a $B$ is a bounded continuous function: $B \in \mathcal{C}(R^2)$. We have the following inclusions of subsets of $\mathcal{D}'(R^2)$:

$$\text{harmonizable} \subset \mathcal{C} \subset L^\infty \subset \mathcal{B}' \subset \mathcal{F}^{-1}(FB) \subset \mathcal{F}^{-1} \text{ (pseudomeasures)}.$$

For none of these classes do we have a simple characterization both of the distributions and of their Fourier transforms (such as the Bochner, Plancherel and Paley-Wiener theorems and their generalizations and other results of Schwartz). Thus which will yield the most useful theory remains unclear.

4. Stochastic processes and random distributions.

**Theorem 4.1.** If $p \geq 1$, a $p$-sub-stationary stochastic process $x(\cdot, \cdot)$ is a $p$-sub-stationary random distribution.

**Proof.** Let $f \in \mathcal{D}(R^k)$. For any $h$ in $R^k$, let

$$A(f, h) = \int \left[ \int_{R^k} f(t - h)x(t, \omega)dt \right]^p dP(\omega)$$

$$= \int \left[ \int_{R^k} f(s)x(s + h, \omega)ds \right]^p dP(\omega).$$

Let $C$ be the support of $f$ and let $\lambda$ be Lebesgue measure. We apply Hölder's inequality to the inner integral, with $q = p/(p - 1)$, obtaining

$$A(f, h) \leq \|f\|_q^p \int_0^\lambda |x(s + h, \omega)|^p d\lambda \leq \|f\|_q^p \lambda(C) \sup_s \int |x(s, \omega)|^p dP(\omega) < \infty.$$

Thus a random distribution $L$ is defined by

$$L(f)(\omega) = \int_{R^k} x(t, \omega)f(t)dt$$
and is $p$-sub-stationary.

For $p < 1$, it seems unclear whether a $p$-sub-stationary stochastic process defines a random distribution at all.

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**REFERENCES**


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