CONTINUITY OF TRANSFORMATIONS WHICH LEAVE INVARIANT CERTAIN TRANSLATION INVARIANT SUBSPACES

BARRY E. JOHNSON
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It is shown that a linear operator \( T: L^2(X) \to L^2(X) \) (\( X \) a locally compact group), with the property that \( TE \subset E \) for each norm closed right translation invariant subspace \( E \) of \( L^2(X) \), is necessarily continuous. In \( \S \) 5 the author shows that this is also true for \( L^1(X) \) when \( X \) contains an element \( a \) which does not lie in any compact subgroup. An example is constructed to show that, in \( l^\infty(\mathbb{R}, \mathbb{R}) \), \( T \) can be discontinuous and still leave invariant each \( \sigma(l^\infty, l^1) \) closed translation invariant subspace of \( l^\infty \). If however \( T: l^\infty(\mathbb{R}, \mathbb{R}) \to l^\infty(\mathbb{R}, \mathbb{R}) \) leaves invariant all norm closed translation invariant subspaces, then \( T \) must be continuous.

We shall use the notation of [3] without further explanation. Sections 3, 4 and 5 overlap with some of the results in 1.4 of Edwards’ paper [3] where he shows inter alia that \( T \) is automatically continuous in the \( L^2 \) case if \( X \) is compact and in the \( L^1 \) case if \( X \) is finitely representable. The work in \( \S \) 6 answers the conjecture in 1.7 of [3].

2. Three basic lemmas. For the first lemma let \( \mathcal{X} \) be a normal topological space and \( \mathcal{A} \) a set of complex valued functions on \( \mathcal{X} \), closed with respect to pointwise multiplication, containing the constant function 1 and such that if \( f \in \mathcal{A} \) then so is \( 1 - f \). We assume that \( \mathcal{A} \) is normal in the sense that if \( F_0 \) and \( F_1 \) are closed disjoint subsets of \( \mathcal{X} \) then there is a function \( f \in \mathcal{A} \) with \( f(F_0) = \{0\}, f(F_1) = \{1\} \). We suppose that there is a mapping \( f \to S_f \) of \( \mathcal{A} \) into \( \mathcal{B}(E) \), where \( E \) is some Banach space, such that \( S_{f_1} = S_f S_{f_1}, S_1 = I, S_{1-f} = I - S_f \). We denote the null space of \( S_f \) by \( N_f \) and suppose that \( T \) is a linear map of \( E \) into \( E \) which leaves invariant all the subspaces \( N_f (f \in \mathcal{A}) \). For each \( f \in \mathcal{A} \) the composition of \( T \) with the quotient map \( E \to E/N_f \) gives a map \( T_f: E \to E/N_f \).

**Definition 2.1.** \( \lambda \in \mathcal{X} \) is called a discontinuity value of \( T \) if \( T_f \) is discontinuous whenever \( f \in \mathcal{A} \) and there exists a neighbourhood \( N \) of \( \lambda \) such that \( f(\lambda') \neq 0 \) for all \( \lambda' \in N \).

**Lemma 2.2.** \( T \) has only a finite number of discontinuity values.

**Proof.** Suppose \( T \) has an infinity of discontinuity values. Then
we can find a sequence \( \lambda_i \) of such values and a closed neighbourhood \( V_i \) of each such that, for each \( i \), \( V_i \) and \( \bigcup_{j \neq i} V_j \) are disjoint. For each \( i \) let \( f_i \) be a function in \( \mathcal{A} \) with \( f_i(V_j) = \delta_{i,j} \) and let \( g_i \in \mathcal{A} \) such that \( g_i(\lambda) = 0 \) for \( \lambda \in V_i \) and \( g_i(\lambda) \neq 0 \) for all \( \lambda \) in some neighbourhood of \( \lambda_i \). Then take \( \xi_i \in E \) such that \( ||\xi_i|| = ||S_{f_i}|| < 2^{-i} \) and \( ||T_{g_i}\xi_i|| > 2^i \). Put \( \eta = \sum S_{g_j}\xi_i \). Then \( S_{g_j}S_{f_i} = 0 \) for \( j \neq i \) so that \( \sum_{i \neq j} S_{f_i}\xi_i \in N_{g_j} \), and \( S_{g_j}S_{f_j} = 0 \) so that \( \xi_j - S_{f_j}\xi_j \in N_{g_j} \). Thus

\[
||T\eta|| \geq ||T_{g_j}\eta|| = ||T_{g_j}S_{f_j}\xi_j|| = ||T_{g_j}\xi_j|| > 2^i
\]

for all \( j \), which is impossible.

**Lemma 2.3.** Let \( A \) be an index set and let, for each \( \alpha \in A \), \( E_\alpha \) be a Banach space and \( P_\alpha \) a continuous linear map from \( E \) into \( E_\alpha \) with null space \( N_\alpha \). Let \( N = \cap N_\alpha \) and let \( P \) be the quotient map \( E \to E/N \). Let \( T \) be a linear map \( E \to E \) such that \( P_\alpha T \) is continuous for each \( \alpha \in A \). Then \( PT \) is continuous.

**Proof.** Suppose \( x_n \to 0 \) in \( E \) and \( PT x_n \to y \) in \( E/N \). Since \( N \subset N_\alpha \) the \( P_\alpha \) give rise to maps \( Q_\alpha : E/N \to E_\alpha \) and \( Q_\alpha PT = P_\alpha T \). Hence \( Q_\alpha y = \lim_n Q_\alpha PT x_n = \lim_n P_\alpha T x_n = 0 \). Thus \( Q_\alpha y = 0 \) for all \( \alpha \in A \) which implies that \( y = 0 \). Hence, by the closed graph theorem, \( PT \) is continuous.

**Lemma 2.4.** Let \( H \) be a Hilbert space, \( \mathcal{A} \) a locally compact Hausdorff space, \( P(\,\cdot\,) \) a spectral measure on \( \mathcal{A} \) (see [5], p. 246), \( \mathcal{G} \) a collection of open sets from \( \mathcal{A} \), \( N = \cap \sigma \mathcal{G} N(P(G)) \) and \( P_0 \) the orthogonal projection with null space \( N \). Then \( P_0 = P(\cup \mathcal{G}) \).

**Proof.** With the usual ordering of projections \( P(\cup \mathcal{G}) \geq P(G) \) for each \( G \) in \( \mathcal{G} \) and so \( P(\cup \mathcal{G}) \geq P_0 \). Now let \( \xi_0 \in N \) and consider the measure \( \mu(Y) = (P(Y)\xi_0, \xi_0) \) defined on the borel sets in \( \mathcal{A} \). Since \( \mu(G) = 0 \) for all \( G \) in \( \mathcal{G} \), \( \cup \mathcal{G} \) is locally of \( \mu \) measure zero and hence since it is open, \( \mu(\cup \mathcal{G}) = 0 \) (see [1], p. 183). However this implies \( P(\cup \mathcal{G})\xi_0 = 0 \) and so \( P_0 = P(\cup \mathcal{G}) \).

3. \( L^2(X) \), the simple case.

**Theorem 3.1.** Let \( X \) contain an element \( a \) which is not contained in any compact subgroup. Let \( T \) be a linear map \( L^2(X) \to L^2(X) \) which leaves invariant each closed right translation invariant subspace of \( L^2(X) \). Then \( T \) is continuous.

**Proof.** Apply Lemma 2.2 with \( \mathcal{A} \) as the circle \( |z| = 1 \) in \( C \) and \( \mathcal{G} \) the set of characteristic functions of measurable sets in \( \mathcal{A} \). The
operator $\tau_a$ is a unitary operator in $L^2(X)$ and its spectral resolution $P(.)$ gives the required map from $\mathcal{A}$ into $\mathcal{B}(L^2(X))$. Since the operators corresponding to elements of $\mathcal{A}$ are weak operator limits of polynomials in $\tau_a$ and $\tau_a^*$, they commute with all right translations. In particular the ranges and null spaces of the values of $P(.)$ are closed right translation invariant subspaces.

In the following it will be convenient not to distinguish too carefully between the quotient of a Hilbert space by a closed subspace and the orthogonal complement of the subspace.

Application of 2.2 shows that $T$ has only finitely many discontinuity values $\lambda_1, \cdots, \lambda_k$. For each $\lambda \in \mathbb{R}$ which is not a discontinuity value we thus have a set $Y \subset \mathbb{R}$ containing $\lambda$ as an interior point such that $P(Y)T$ is continuous; clearly $Y$ can be taken to be open. Application of 2.3 taking $A$ as the class of open sets $G$ in $\mathbb{R}$ with $P(G)T$ continuous and $P_\sigma = P(G)$ shows that $P_\sigma T$ is continuous where $P_\sigma$ is the orthogonal projection along the intersection of the null spaces of the $P(G); G \in A$. Lemma 2.4 then shows that $P_\sigma = I - P(\{\lambda_1\}) - \cdots - P(\{\lambda_k\})$.

If now $\xi \neq 0$ is in the range of $P(\{\lambda_i\})$ then $\tau_a \xi = \lambda_i \xi$. We can find a compact set $K$ such that $\int_K |\xi(x)|^2 dx > 1/2, ||\xi||^2$ and an integer $n$ such that $a^n \in KK^{-1}$. Thus

$$1/2, ||\xi||^2 < \int_K |\xi(x)|^2 dx = \int_K |\lambda_i^n \xi(x)|^2 dx = \int_K |\xi(a^{-n}x)|^2 dx = \int_{a^{-n}K} |\xi(x)|^2 dx.$$

Since $a^{-n}K \cap K$ is void this gives

$$||\xi||^2 \geq \int_K |\xi(x)|^2 dx + \int_{a^{-n}K} |\xi(x)|^2 dx > 1/2, ||\xi||^2 + 1/2, ||\xi||^2.$$

This contradiction shows that the range of $P(\{\lambda_i\})$ is the zero subspace and $P(\{\lambda_i\}) = 0$. Similarly $P(\{\lambda_i\}) = \cdots P(\{\lambda_i\}) = 0, P_\sigma = I$ and $T$ is continuous.

**Remark 3.2.** A stronger result would be obtained if we could consider operators reduced by subspaces closed in some topology weaker than the norm topology. Since $L^2$ is reflexive there is no advantage in considering the weak topology. The result is untrue if we take the topology $\sigma(L^2, L_0)$ where $L_0$ is the subspace of $L^2$ consisting of
functions with compact support since, in the case $X = \mathbb{Z}$, the additive group of the integers, there are no $\sigma(L^2, L_0)$-closed translation invariant proper subspaces of $L^2$. In this case if $\xi \in L_0, \xi \neq 0$ then for $\eta$ and all its translates to be in $\xi^\perp$ we must have $\xi^\ast \eta = 0$, or $\hat{\xi} \hat{\eta} = 0$ where $\hat{\xi}, \hat{\eta}$ are the functions in $L^2$ of the unit circle whose Fourier coefficients are $\xi, \eta$. Since $\hat{\xi}$ is a polynomial this implies that $\hat{\eta} = 0$ almost everywhere and so $\eta = 0$.

**Remark 3.3.** If $X$ is not unimodular or has a nonunimodular subgroup then it satisfies the condition of the theorem.

4. $L^2(X)$, the general case. The methods of [3] apply when the group $X$ is finitely representable. To show that there is indeed ground not covered by either Edwards' method or Theorem 3.1 we take the group in the following example with the discrete topology.

**Example 4.1.** The alternating group $A$ on $\xi$ symbols (see [4], p. 40) is not finitely representable yet every element lies in a finite subgroup.

**Proof.** We consider the permutations as acting on $1, 2, 3, \cdots$ and denote by $A_n$ the subgroup leaving $n + 1, n + 2, \cdots$ invariant. $A_n$ is isomorphic with the alternating group on $n$ symbols in an obvious way. Suppose $\varphi: A \rightarrow GL(m)$ is an $m$ dimensional representation of $A$. Then $\varphi$ induces $m$ dimensional representations of $A_n$ for each $n$ which must be trivial for $n > m + 1$ (see [2], p. 466). Since $A = \cup A_n, \varphi$ is trivial on $A$. Thus there is no nontrivial finite dimensional representation of $A$.

**Theorem 4.2.** Let $X$ be locally compact but not compact. Let $T$ be a linear map $L^2(X) \rightarrow L^2(X)$ which leaves invariant each closed right translation invariant subspace of $L^2(X)$. Then $T$ is continuous.

**Proof.** We use here ideas occuring in the theory of $W^*$-algebras (see e.g. [5], §41).

Let $\mathcal{Q}$ be the set of elements of $\mathcal{B}(L^2)$ commuting with all the right translation operators. The range and null space of any projection in $\mathcal{Q}$ is closed with respect to right translation and hence each such projection commutes with $T$. We consider $Z$, the centre of $\mathcal{Q}$, a commutative $W^*$-algebra. Let $\mathcal{M}$ be its maximal ideal space so that $\mathcal{M}$ is an extremally disconnected compact topological space (see [6], p. 294). To show that $T$ has only finitely many discontinuity values $\lambda_1, \cdots, \lambda_k$, we then apply Lemma 2.2 with $\mathcal{M}$ as $\mathcal{K}, \mathcal{M}$ as the set of characteristic functions of open closed sets in $\mathcal{K}$ and for $f \in \mathcal{M}$,
CONTINUITY OF TRANSFORMATIONS 227

Sf as the projection in Z whose Gelfand transform is f.

Applying 2.3 with A as the set of open closed sets G in \( \mathcal{M} \) for which \( P(G)T \) is continuous, where \( P(G) \) is the element of \( Z \) whose Gelfand transform is \( f \). From the way in which it is constructed, \( P \in Z \), and so \( P = P(G_0) \) where \( G_0 \) is the maximum element in A which must be \( \mathcal{M} \{ \lambda_1, \ldots, \lambda_k \} \). This shows that \( \{ \lambda_1, \ldots, \lambda_k \} \) is open closed and, since \( \mathcal{M} \) is Hausdorff, that if \( T \) is discontinuous then \( \mathcal{M} \) has isolated points.

Let now \( \lambda_0 \in \{ \lambda_1, \ldots, \lambda_k \} \) and let \( H_0 \) be the range of the corresponding projection \( P_0 = P(\{\lambda_0\}) \).

The algebras \( \mathfrak{L}_0, \mathfrak{L}_* \) formed by restricting the operators in \( \mathfrak{L}, \mathfrak{L}' \) to \( H_0 \) form a factored pair (see [5], p. 450). Now let \( C \) be a maximal commutative subalgebra of \( \mathfrak{L}_0 \beta \). Application of the argument in the previous paragraphs to \( C \) in place of \( Z \) shows that the maximal ideal space of \( C \) contains at least one isolated point \( \mu \). Let \( M_0 \) be the range of the corresponding projection \( Q \). Since \( Q \in \mathfrak{L}_0, M_0 \) is a subspace adjoined to \( \mathfrak{L}_0 \) and is moreover a minimal subspace ([5], p. 451). It follows that the factor \( \mathfrak{L}_0 \) is of type I. The structure of such factors is as follows ([5], p. 480). \( H_0 \) is a direct sum of orthogonal subspaces \( M_{\alpha}; \alpha \in A \), each isomorphic with \( M_0 \) (and which we shall consider as copies of \( M_0 \)). Relative to this decomposition of \( H_0 \), the matrix of an element of \( \mathfrak{L}_0 \) is \( [a_{\alpha\beta}I_0] \) where \( a_{\alpha\beta} \) are complex numbers and \( I_0 \) is the identity operator in \( M_0 \). This is an isomorphism between \( \mathfrak{L}_0 \) and those matrices of the form \( [a_{\alpha\beta}I_0] \) giving bounded transformations.

Applying the results in § 2 a third time we see that there is only a finite number of values \( \alpha_1, \ldots, \alpha_n \) of \( \alpha \) for which the restriction of \( T \) to \( M_{\alpha} \) is not continuous. Let, if possible, \( \alpha_0 \) be a value of \( \alpha \) different from \( \alpha_1, \ldots, \alpha_n \). Let \( P \) be the operator in \( \mathfrak{L}_0 \) whose matrix \( [a_{\alpha\beta}I_0] \) is given by \( a_{\alpha\beta} = \alpha_{\alpha_0\alpha} = 1, a_{\alpha\beta} = 0 \) otherwise. \( P \) is a non-orthogonal projection and \( PT_0 = T_0P \) (where \( T_0 \) is the restriction of \( T \) to \( H_0 \)) so that \( PT_0M_{\alpha_1} = T_0PM_{\alpha_1} = T_0M_{\alpha_0} \) which is impossible since \( P \) maps \( M_{\alpha_1} \) isometrically onto \( M_{\alpha_0} \) whilst \( T_0 \) is bounded on \( M_{\alpha_0} \) and unbounded on \( M_{\alpha_1} \). Thus \( A = \{ \alpha_1, \ldots, \alpha_n \} \).

Now let \( \xi_0 \in M_0 \) with \( ||\xi_0|| = 1 \) and let \( \xi_1, \ldots, \xi_n \) be its copies in \( M_{\alpha_1}, \ldots, M_{\alpha_n} \). The subspace of \( L(X) \) generated by \( \xi_1, \ldots, \xi_n \) is closed with respect to all operators in \( \mathfrak{L} \). Let \( K \) be a compact set in \( X \) such that

\[
\int_K |\xi_i(t)|^2 \, dt > 1 - (2n)^{-\delta}
\]

for \( i = 1, \ldots, n \) and let \( a \in X\setminus KK^{-1} \) (such an element exists since \( KK^{-1} \) is compact and \( X \) is not). Then \( \tau_{a,\xi_1} = a_1\xi_1 + \cdots + a_n\xi_n \) where
1. Let \( c \) be the characteristic function of \( aK \). Then

\[
\| (\tau \xi) c \|^2 = \int_X |\xi_\tau(t)|^2 \, dt > 3/4
\]

whereas

\[
\| \xi_\tau c \|^2 \leq \int_X |\xi_\tau(t)|^2 \, dt \leq (2n)^{-1}
\]

so that

\[
\| (\tau \xi) c \| = \| a_1 \xi_1 c + \cdots + a_n \xi_n c \|
\leq |a_1| \| \xi_1 c \| + \cdots + |a_n| \| \xi_n c \|
\leq 1/2
\]

which is a contradiction, proving the theorem.

5. \( L'(X) \), the simple case.

**Theorem 5.1.** Let \( X \) contain an element \( a \) which is not contained in any compact subgroup. Let \( T \) be a linear operator from \( L'(X) \) into \( L'(X) \) such that \( \mu^* T f = 0 \) whenever \( \mu \in M_w(X) \), \( f \in L'(X) \) with \( \mu^* f = 0 \). Then \( T \) is continuous.

**Proof.** As in 3.1 take \( \mathfrak{X} \) as the circle \( |z| = 1 \) and \( \mathcal{A} \) as the set of functions \( f \) on \( \mathfrak{X} \) with absolutely convergent Fourier series \( \sum a_n e^{i n \theta} \). We take for \( S_f \) the operator \( g \mapsto \mu_f^* g \) where, for \( f \in \mathcal{A} \), \( \mu_f = \sum a_i (\delta_{\theta_i})^i \). By Lemma 2.2, \( T \) has only finitely many discontinuity values \( \lambda_1, \ldots, \lambda_n \). Let \( \mathcal{A}_0 \) be the subset of \( \mathcal{A} \) consisting of those elements \( f \) for which \( T_f \) is continuous. For each \( \lambda \in \mathcal{X} \) different from \( \lambda_1, \ldots, \lambda_n \) there is \( f \in \mathcal{A}_0 \) such that \( T_f \) is continuous and \( f(\lambda) \neq 0 \). Put \( N = \cap N_f \ (f \in \mathcal{A}_0) \). If \( \{\beta_j\} \in \ell(-\infty, +\infty) \) with \( \beta \tau f = 0 \) for all \( f \in \mathcal{A}_0 \) where \( \beta = \sum \beta_j z^j \) then \( \beta_j = 0 \) for all \( j \) and using the fact that \( \mathcal{X} \{\lambda_1, \ldots, \lambda_n\} \) is \( \sigma \)-compact, we can find a countable subset \( \mathcal{A}_a \) of \( \mathcal{A}_0 \) with this property. Now if \( g \in N \), then for all \( \sum a_n z^n \in \mathcal{A}_a \), \( (\mu^* g)(t) = \sum a_n g(a^{-k}t) = 0 \) for almost all \( t \) in \( X \). Putting \( \beta_{a,k} = g(a^k t), \ k = -1, 0, 1, \ldots \) we have, for each \( f \in \mathcal{A}_a \), \( \beta \tau f = 0 \) for almost all \( t \) in \( \mathcal{X} \). Since \( \mathcal{A}_a \subset \mathcal{A}_0 \) and \( \mathcal{A}_a \) is countable, we have that for almost all \( t \) in \( \mathcal{X} \), \( \beta \tau f = 0 \) for all \( f \in \mathcal{A}_a \). Thus \( \beta \tau = 0 \) for almost all \( t \) in \( \mathcal{X} \) and, in particular, \( \beta_{a,k} = g(t) = 0 \) a.e. in \( \mathcal{X} \) so that \( N = \{0\} \). Using 2.3 with \( A = \mathcal{A}_0 \) and \( P_f \) as the quotient map \( L'(X) \to L'(X)/N_f \), the proof is complete.
REMARK 5.2. The author has not been able to extend this result to the general case.

REMARK 5.3. The condition $\mu^* T f = 0$ whenever $\mu^* f = 0$ is of course equivalent to saying that the spaces $N_\mu = \{ f; f \in L^*(X), \mu^* f = 0 \}$ are closed with respect to $T$. These spaces are right translation closed and also closed in a variety of topologies including $\sigma(L^1, C_0)$ or any other topology $\sigma(L^1, F)$ where $F$ is a subspace of $L^\infty$ total over $L^1$ and closed with respect to convolution with bounded measures.

6. The space $l^\infty$. We now show that the continuity hypothesis in 1.6 of [3] is necessary in the case $X = \mathbb{Z}$ (the additive group of integers). We use the results in Ch. VII, § 1 of [7].

EXAMPLE 6.1. The elements $a = \{ a_n \}$ of $l^\infty(-\infty, +\infty)$ can be considered as Fourier coefficients of certain distributions $\hat{a}$ on the circle group $\mathbb{Z}$ and since $\{ a_n (1 + n^2)^{-1} \} \in \ell(-\infty, +\infty)$ these distributions are, at worst, of order two. If $b = \{ b_n \}$ is a sequence of rapid decrease it is the sequence of Fourier coefficients of an infinitely differentiable function $\hat{b}$ on the circle and we have $a^* b \in l^\infty, (a^* b)^{-} = \hat{a} \cdot \hat{b}$. It follows from this and the ideas of spectral synthesis, that the constant sequence 1 lies in the $\sigma(l^\infty, l^1)$-closed linear subspace of $l^\infty$ spanned by the translates of $a$ if and only if 0 lies in the support of $\hat{a}$. For each positive integer $n$ let $E_n = \{ a; a \in l^\infty, d(0, \text{supp } \hat{a}) \geq 1/n \}$. The $E_n$ are norm closed subspaces of $l^\infty$, $E_0 \subseteq E_{n+1}$ so that $E_\infty = \bigcup E_n$ is of first category with the induced $l^\infty$ topology and so is not closed in $l^\infty$. We can thus find a discontinuous functional $f$ on $l^\infty$ which is 0 on $E_\infty$. If we then put $T a = f(a) 1$ we then have a discontinuous linear transformation $T$ on $l^\infty$ such that $T a$ is in the $\sigma(l^\infty, l^1)$-closure of linear combinations of translates of $a$.

6.2. The conjecture in 1.7 of [3] can only break down in a way similar to that in 6.1. This can be seen by application of the results in § 2 taking $\mathcal{F}$ as $|z| = 1$ and $\mathcal{S}$ as the set of infinitely differentiable functions on $\mathbb{R}$. We see that if $T$ is a linear transformation $l^\infty \rightarrow l^\infty$ such that $T a$ is a $\sigma(l^\infty, l^1)$-limit of linear combinations of translations of $a$ then there is a subspace $N$ such that $P T$ is continuous where $P$ is the quotient map $l^\infty \rightarrow l^\infty/N$. The space $N$ is the intersection of null spaces corresponding to a subset $\mathcal{S}_0$ of $\mathcal{S}$, the elements of $\mathcal{S}_0$ having only a finite number of common zeros $\lambda_1, \ldots, \lambda_n$. Thus $N$ consists only of elements $a$ with $\text{Supp } \hat{a} \subset \{ \lambda_1, \ldots, \lambda_n \}$ and since $a$ is of order $\leq 2$ we have

$$a_n = \alpha_1 \lambda_1^n + \cdots + \alpha_n \lambda_n^n + n(\beta_1 \lambda_1^n + \cdots).$$
The coefficient of $n$ is almost periodic so that for $a$ to be bounded the $\beta_i$ are all zero. Hence $N$ is a space of linear combinations of the $\lambda_j, j = 1, \cdots, l$.

6.3. If $T$ is assumed to leave invariant all norm closed translation invariant subspaces of $l^\omega$ then it is continuous. To show this we apply the argument in 6.2 and then put

$$l^+ = \{a; a \in l^\omega, a_n = 0 \text{ for } n < 0\}$$

$$l^- = \{a; a \in l^\omega, a_n = 0 \text{ for } n \geq 0\}.$$  

Let $K$ be the norm of $PT$. If $a \in l^\omega$ then $a = a^+ + a^-$ with $a^+ \in l^+$, $a^- \in l^-$. Then $||PTa^+|| \leq K ||a^+|| \leq K ||a||$ and so there is $b \in N$ such that $||Ta^+ - b|| \leq 2K ||a||$. However since $Ta^+$ is a uniform limit of linear combinations of translates of $a^+$, and hence $(Ta^+)_n \rightarrow 0$ as $n \rightarrow -\infty$, and $b$ is almost periodic we have $||b|| \leq ||Ta^+ - b||$. From this we see that $||Ta^+|| \leq ||Ta^+ - b|| + ||b|| \leq 4K ||a||$, and applying the same argument to $||Ta^-||$ we get a similar conclusion, the two together giving $||Ta|| \leq 8K ||a||$.

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Edward Dewey Davis, *Ideals of the principal class, R-sequences and a certain monoidal transformation* .................................................. 197
Richard Mansfield Dudley, *Sub-stationary processes* .................. 207
Newton Seymour Hawley and M. Schiffer, *Riemann surfaces which are doubles of plane domains* .................................................. 217
Barry E. Johnson, *Continuity of transformations which leave invariant certain translation invariant subspaces* .......................... 223
John Eldon Mack and Donald Glen Johnson, *The Dedekind completion of $C(\mathcal{X})$* ................................................................. 231
H. D. Miller, *Generalization of a theorem of Marcinkiewicz* ....... 261
Joseph Baruch Muskat, *Reciprocity and Jacobi sums* ................. 275
Stelios A. Negrepontis, *On a theorem by Hoffman and Ramsay* .... 281
Paul Adrian Nickel, *A note on principal functions and multiply-valent canonical mappings* ..................................................... 283
Robert Charles Thompson, *On a class of matrix equations* ........ 289
David Morris Topping, *Asymptoticity and semimodularity in projection lattices* ........................................................................ 317
James Ramsey Webb, *A Hellinger integral representation for bounded linear functionals* ..................................................... 327
Joel John Westman, *Locally trivial $C^r$ groupoids and their representations* ............................................................. 339
Hung-Hsi Wu, *Holonomy groups of indefinite metrics* .............. 351