DECOMPOSITIONS OF $E^3$ WHICH YIELD $E^3$

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In recent years interest has been focused on the following two questions.

If $G$ is an upper semi-continuous decomposition of $E^3$ whose decomposition space $G'$ is homeomorphic to $E^3$, under what conditions can we conclude that

1. each element of $G$ is point-like?
2. there is a pseudo-isotopy $F: E^3 \times [0,1] \to E^3$ such that $F|E^3 \times 0$ is the identity and $F|E^3 \times 1$ is equivalent to the projection map $\Pi: E^3 \to G'$?

An example of Bing of a decomposition of $E^3$ into points, circles, and figure-eights shows that some additional hypotheses must be inserted. The theorem presented here gives such hypotheses, namely that the nondegenerate elements form the intersection of a decreasing sequence of finite disjoint unions of cells-with-handles, and project into a Cantor set.

For definitions and notation see [1]. In the example of Bing previously mentioned, the image of the union of the nondegenerate elements $H^*$ under the projection map $\Pi$ is an arc. Thus, the first condition one might impose in an attempt to answer the above questions is that $\Pi(H^*)$ be a Cantor set. I suspect that this is sufficient, however, that is still unknown. We use an additional hypothesis here.

**Theorem.** Let $G$ be an upper semi-continuous decomposition of $E^3$ whose decomposition space $G'$ is $E^3$ and let the image $\Pi(H^*)$ of the union of all the nondegenerate elements be a Cantor set. Suppose also that $G$ is definable by cells-with-handles, that is

$$H^* = \bigcap_{i=1}^{\infty} \left( \bigcap_{j=1}^{N_i} C_{ij} \right)$$

where each $C_{ij}$ is a cell-with-handles, $C_{ij} \cap C_{ik} = \emptyset$ for $j \neq k$, and $\bigcup_{i=1}^{N_i} C_{ij}$ is contained in the point-set interior of $\bigcup_{j=1}^{N_{i-1}} C_{i-1,j}$ for $i = 2, 3, \ldots$. Then each element of $G$ is point-like and there is a pseudo-isotopy $F: E^3 \times [0,1] \to E^3$ such that $G = \{F^{-1}(x, 1)\}_{x \in E^3}$.

**Proof of the theorem.** By Bing’s approximation theorem, we can assume that each $C_{ij}$ is polyhedral. We will rely on the following theorem of Hempel [3].

**Theorem (Hempel).** Suppose $C$ and $C'$ are polyhedral 3-manifolds with boundary in $S^3$ such that $C$ is a cell-with-handles and such...
that there is a map $f$ of $C$ onto $C'$ which takes $\text{Bd}(C)$ homeomorphically onto $\text{Bd}(C')$. Then $C$ and $C'$ are homeomorphic; in particular, $f|_{\text{Bd}(C)}$ can be extended to a homeomorphism of $C$ onto $C'$.

We will first show that if $g \in G$, then $g$ is point-like. Let $U$ be some neighborhood of $g$. Then some $C_{ij}$ of the theorem is such that $g \subset \text{Int} C_{ij} \subset C_{ij} \subset U$. We will find a cell $C$ such that $g \subset \text{Int} C \subset C_{ij}$. $II(g)$ is a point and $II(g) \in \text{Int} II(C_{ij})$ so there is a cell $C'$ such that $II(g) \in \text{Int} C' \subset II(C_{ij})$. For this fixed $C'$ there must be an $i'$ and a $j'$ such that $II(g) \in II(C'_{i'j'}) \subset \text{Int} C'$. For each $k = 1, 2, \cdots, j', \cdots, N_i$, we will modify $II$ on $C'_{i'k}$ so that the new map $II'$ is a homeomorphism except on $C'_{i'j'}$. We can do this because of Hempel's theorem. It is then easy to show that $II'^{-1}(C')$ is the cell $C$ we are seeking, since $II'^{-1}$ is a homeomorphism on $\text{Bd} C'$.

In order to prove that $G'$ may be realized by pseudo-isotopy we need only show the following lemma is true. The theorem will then follow [2].

**Lemma.** If $G$ is as in the theorem, $\varepsilon > 0$ is given, and $U$ is any neighborhood of $H^*$, then there is an isotopy $F: E^3 \times [0, 1] \to E^3$ such that $F|_{E^3 \times 0} = 1$, $F(x, t) = x$ for all $x \in E^3 - U$, $t \in [0, 1]$, and for each $g \in G$, $F(g(1))$ has diameter less than $\varepsilon$.

**Proof.** There is an $i$ such that $C_{ij} \subset U$ for $j = 1, \cdots, N_i$. We will take $F(x, t) = x$ for all $x \in E^3 - \bigcup_{l=1}^{N_i} C_{ij}$. For each $j$ there is a homeomorphism $h_j: II(C_{ij}) \to C_{ij}$ which agrees with $II^{-1}$ on the boundary and we will define a map $II': E^3 \to E^3$ as follows. For all $x \in E^3 - \bigcup_{l=1}^{N_i} C_{ij}$ let $II'(x) = x$. For $x \in C_{ij}$ let $II'(x) = h_j II(x)$. There is an integer $k$ such that $II'^{-1}(C_{kl})$ has diameter less than $\varepsilon$ for each $l = 1, 2, \cdots, N_k$. We may also assume that $II'$ is piecewise linear on $E^3 - \bigcup_{l=1}^{N_k} \text{Int} C_{kl}$. Using Hempel's result again we modify $II'$ on each $C_{kl}$ so that the new map $II''$ is a piecewise linear homeomorphism agreeing with $II'$ everywhere except in $\bigcup_{l=1}^{N_k} \text{Int} C_{kl}$. Note that for each $g \in G$, $\text{diam } II''(g) < \varepsilon$. The proof is completed by the following lemma.

**Lemma.** Let $C$ be a polyhedral cell-with-handles in $E^3$ and let $h$ be a piecewise linear homeomorphism of $E^3$ onto itself such that $h|_{\text{Int } C}$ is the identity. Then $h|_{C}$ is isotopic to the identity.

**Proof of lemma.** This lemma appears to be well known, however, an outline of the proof is included for completeness. Since $C$ is a polyhedral cell will-handles, there is a collection of mutually disjoint polyhedral disks $D_1, \cdots, D_n$ with $D_i \cap \text{Bd} C = \text{Bd} D_i$, $\text{Int } D_i \subset \text{Int } C$ and...
such that $C$ is the union of two cells $C_i$ and $C_j$ whose intersection is $\bigcup_{i=1}^{n} D_i$. Since $h(D_i)$ is polyhedral and $h(D_i) \cap \text{Bd} \ C = D_i \cap \text{Bd} \ C$ there is an isotopy $H: C \times [0, 1] \to C$ with $H(x, 0) = h(x)$ and $H(x, t) = x$ for all $x \in \text{Bd} \ C$ and $t \in [0, 1]$ and $H(x, 1) = x$ for $x \in \bigcup_{i=1}^{n} D_i$. Then $H: C \times 1 \to C$ is a homeomorphism of $C$ onto itself which is the identity on $\text{Bd} \ C_i \cup \text{Bd} \ C_j$ and we may find the appropriate isotopy returning $H: C \times 1 \to C$ to the identity.

*Question.* In the theorem is the requirement that each $C_{ij}$ is a cell-with-handles necessary? Certainly since the image of the union of the nondegenerate elements is a Cantor set in $E^3$, it has this cell-with-handles intersection property. It is true that a 3-manifold-with-boundary need not be a cell-with-handles in order to map onto a cell-with-handles with a map which is a homeomorphism on the boundary; however, I believe that these maps would have to have a continuum of nondegenerate elements. The maps we are considering have only a Cantor set of nondegenerate elements.

**References**


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