

Pacific Journal of Mathematics

**INEQUALITIES FOR FUNCTIONS REGULAR AND BOUNDED
IN A CIRCLE**

CECIL CRAIG, JR. AND A. J. MACINTYRE

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This paper is concerned with functions $w = f(z)$ regular and satisfying the inequality $|f(z)| < 1$ in $|z| < 1$. This class of functions will be denoted E .

We determine conditions on z_1, z_2, z_3 and w_1, w_2, w_3 for

$$w_k = f(z_k) \quad (k = 1, 2, 3)$$

to be possible with an $f(z)$ of E . In particular to map the vertices of the equilateral triangle $z_k = re^{2k\pi i/3}$ into the vertices of another taken in the opposite direction $w_k = \rho e^{-2k\pi i/3}$ we must have $\rho \leq r^2$. The extremal function associated with this problem is $w = z^2$. It seems convenient to discuss the fixed point if any of the mapping of $|z| < 1$ into $|w| < 1$. We include a simple proof of the theorem of Denjoy and Wolf that if no such fixed point exists then there is a unique distinguished fixed point on $|z| = 1$. We give several results restricting the position of the interior or distinguished boundary fixed point in terms of the location of a zero of $f(z)$ or the value $f(0)$.

The theorem of Pick asserts that if $f(z)$ is in E then $D(f(z_1), f(z_2)) \leq D(z_1, z_2)$ where the nonEuclidean distance

$$D(z_1, z_2) = \frac{1}{2} \log \frac{1 + d(z_1, z_2)}{1 - d(z_1, z_2)} \quad \text{with} \quad d(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|.$$

Equality holds if and only if f sets up a Möbius transformation. It follows from Pick's theorem that there can be at most one fixed point of $w = f(z)$ in $|z| < 1$ unless $f(z) \equiv z$. It is usually sufficient when f has an interior fixed point at $z = \alpha (\neq 0)$ to suppose $0 < \alpha < 1$.

Our first four theorems give information about the relative positions of zeros of f , an interior fixed point, and the value $f(0)$. We exclude the case where $f(z) \equiv z$.

THEOREM 1. *Let $f \in E$ and $f(0) \neq 0$. Then f has no zeros in $|z| < |f(0)|$; and has a zero on $|z| = |f(0)|$ if and only if f determines a Möbius transformation.*

Proof. The image of $|z| \leq |f(0)|$, which we denote by C under the transformation $w = (z + f(0))/(1 + \bar{f}(0)z)$ is a circular disc C' having nonEuclidean center $f(0)$ with boundary passing through the origin. The function $w = f(z)$ takes the closed disc C inside C' in the case f

is not a Möbius transformation so that $f(z) \neq 0$ for $z \in C$. If f is linear, nonEuclidean distances are preserved and $f(0)$ is on the boundary of C' .

THEOREM 2. *Let $f \in E$ and let $z = \alpha$ be a fixed point of f with $0 < \alpha < 1$. Then f has no zeros inside $|z - \alpha/(1 + \alpha^2)| = \alpha/(1 + \alpha^2)$ and has a zero on the boundary if and only if f determines a Möbius transformation.*

Proof. The conclusion follows directly from Pick's theorem since the circle described is the nonEuclidean circle with nonEuclidean center $z = \alpha$.

If $f(0)$ is known in addition to the existence of an interior fixed point $\alpha (\neq 0)$, then these two results can be combined to give a larger region which is zero-free, namely the union of the two closed discs. The boundary zero of f occurs at $z = \overline{f(0)}$ when f is a Möbius transformation.

THEOREM 3. *If $f \in E$ and $f(0) \neq 0$, then there can be no fixed point interior to the circle $C_1: |z| = (1 - \sqrt{1 - |f(0)|^2})/|f(0)|$; and a fixed point on the boundary at $z_0 = e^{i \arg f(0)} (1 - \sqrt{1 - |f(0)|^2})/|f(0)|$ only if f determines a Möbius transformation.*

Proof. The nonEuclidean midpoint of the segment from 0 to $f(0)$ is z_0 (See Figure 1). A displacement of all points inside C_1 by $w = f(z)$ insures there can be no fixed point interior to C_1 . The boundary case is clear.

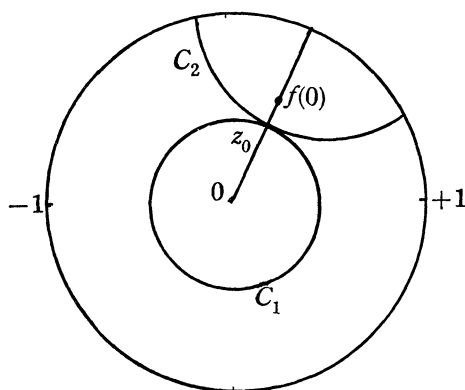


Figure 1.

If f is known to have an interior fixed point, an improvement over Theorem 3 can be made as to its location based on a knowledge of $f(0)$. This is indicated in:

THEOREM 4. *Let $f \in E$. If f has a fixed point $z = \alpha (\neq 0)$, in $|z| < 1$, then α lies inside the circle C_2 passing through z_0 (Figure 1) with center at the geometric inverse of $f(0)$, relative to the unit circle and is on the boundary if and only if f sets up an elliptic Möbius transformation.*

Proof. This is a direct consequence of the inequality $D(\alpha, 0) \geq D(\alpha, f(0))$ where the point $z = \alpha$ is considered variable and $f(0)$ is fixed. The assumed interior fixed point is nearer $f(0)$ than the origin in the nonEuclidean sense, except when f is linear. This requires an investigation of the nonEuclidean perpendicular bisector of the radial segment from 0 to $f(0)$. Straight lines of the Poincaré model are Euclidean circles orthogonal to the unit circle. The Euclidean circle C_2 passing through the point z_0 and orthogonal to $|z| = 1$ is the one described in the statement of the theorem.

Theorem 4 provides a simple proof of the Theorem of Denjoy on the fixed points of analytic transformations of the unit circle into itself [3]. It is convenient to develop the argument by formulating several variants of Theorem 4. $f(z)$ is supposed to belong to E .

THEOREM 4 A. *If $f(0) \neq 0$ and $\arg f(0) = \theta$, then any interior fixed point must lie in the half plane $R(e^{i\theta}z) > 0$.*

The half plane evidently contains the circle C_2 of Theorem 4.

THEOREM 4 B. *The nonEuclidean bisector of the nonEuclidean segment joining x and $f(x)$ divides the unit circle into two parts. Any interior fixed point must lie in the part containing $f(x)$ unless the function sets up an elliptic linear transformation when the fixed point must lie on the bisector.*

This statement is equivalent to that of Theorem 4. We have only to apply Theorem 4 to $w = Tf(T^{-1}z)$ where T is a linear transformation of $|z| < 1$ into itself which carries x to the origin.

THEOREM 4 C. *If x and $f(x)$ have the same argument, then any interior fixed point must lie on the same side as $f(x)$ of the circle through x and orthogonal to the radius Ox and to $|z| = 1$.*

This follows from Theorem 4 A. We consider $w = Tf(T^{-1}z)$ where T carries x to the origin and the diameter through x into itself.

Now consider $z = g(w)$ the solution of $wf(z) = z$. From Rouché's theorem $g(w)$ is regular and one valued for w in $|w| < 1$. Let $0 < w < w' < 1$. Apply Theorem 4 C to $F(z) = w'f(z)$. Let $\alpha = g(w)$. We know that $F(\alpha) = w'\alpha/w$. Any fixed point of $F(z)$ and that is to say $g(w')$ must lie in the smaller part of the unit circle partitioned

as in Theorem 4 C. If $g(w)$ does not tend to a fixed point of $|w| < 1$ as $w \rightarrow 1$ by positive values, it must converge to a point of $|z| = 1$. This point on $|z| = 1$ is the Denjoy distinguished fixed point. Calling such points D fixed points it is clear that Theorem 4 applies to these as well as to interior fixed points.

We shall next be concerned with special cases of three point interpolation by $f \in E$. The problem first considered is that in which we require the vertices of an isosceles triangle to be mapped by f into vertices of another isosceles triangle.

THEOREM 5. *A necessary and sufficient condition for the existence of a function $f \in E$ taking points $z_0, 0, \bar{z}_0$ into $w, 0, \bar{w}$, respectively, is that $w = f(z_0)$ lies in lens $B = \{t \mid t = z_0\zeta, \zeta \in A\}$, where A is the lens formed by the two circular arcs passing through $-1, z_0, +1$ and $-1, \bar{z}_0, +1$.*

Proof. This follows from an inequality of G. Julia [4, 74-78] which for our problem is expressed by $D(w/z_0, \bar{w}/\bar{z}_0) \leq D(z_0, \bar{z}_0)$. Since D is a monotone increasing function of d , it is sufficient for our purpose to use d and we shall refer to this as the nonEuclidean distance.

Let $\delta = |(z_0 - \bar{z}_0)/(1 - z_0^2)|$ and $\zeta = w/z_0 = x + iy$. Then the basic inequality becomes $|(\zeta - \bar{\zeta})/(1 - \zeta^2)| \leq \delta$ or $2|y|/|1 - (x + iy)^2| \leq \delta$. On squaring and simplifying we have $4y^2(1 - \delta^2) \leq \delta^2(1 - \{x^2 + y^2\})^2$. After taking square roots and rearranging we obtain

$$x^2 + \left(|y| + \frac{\sqrt{1 - \delta^2}}{\delta}\right) \leq \left(\frac{1}{\delta}\right)^2.$$

If $y \geq 0$, ζ lies on or below one circular arc; for $y < 0$, ζ lies on or above the other arc, the reflection of the first in the real axis. These arcs form the boundary of a lens. To see that the boundary curves pass through z_0 and \bar{z}_0 , consider the case of equality $|\zeta - \bar{\zeta}|/|1 - \zeta^2| = |z_0 - \bar{z}_0|/|1 - z_0^2|$. This equation describes the locus of a point which is a fixed nonEuclidean distance from its conjugate, in this case the nonEuclidean distance being $d(z_0, \bar{z}_0)$. The lens just described is labeled A in Figure 2. To complete the proof one notes that $w = z_0\zeta$, for $\zeta \in A$, is the set of points of lens B .

A slightly more general result than Theorem 5 can be obtained. We require f to be real at a real point h as well as to take conjugate values at the conjugate pair z_0, \bar{z}_0 .

THEOREM 6. *A necessary and sufficient condition for the existence of a function $f \in E$ taking z_0, h, \bar{z}_0 into w, h', \bar{w} , respectively, where h and h' are real numbers, is that $w = f(z_0)$ lies in a lens*

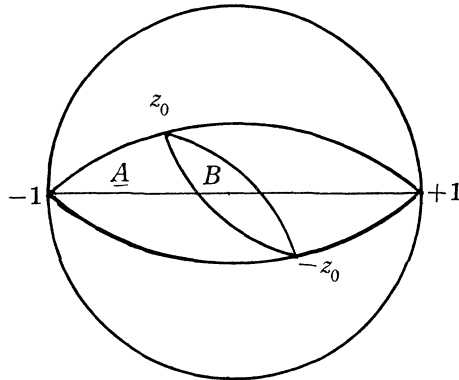


Figure 2.

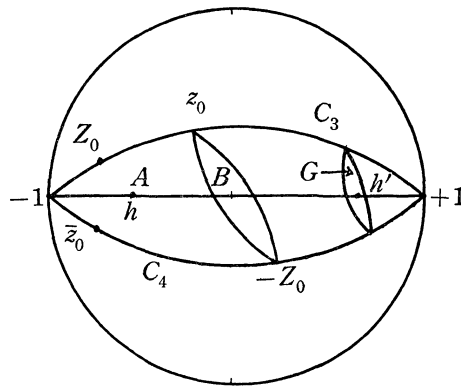


Figure 3.

$G = \{t \mid t = (W + h')/(1 + h'W), W \in B\}$, where

$$B = \left\{ W \mid W = Z_0 \zeta, \zeta \in A, Z_0 = \frac{z_0 - h}{1 - h z_0} \right\}$$

and A is the lens described in Theorem 5.

Proof. The proof depends on the fact that the composition of functions in E is again in E . The transformation $Z = (z - h)/(1 - hz)$ takes h to zero with z_0 and \bar{z}_0 going to conjugate points $Z_0 = (z_0 - h)/(1 - h z_0)$ and \bar{Z}_0 . Since this transformation preserves nonEuclidean distances, z_0 is moved to Z_0 on circular arc C_3 which passes through $-1, z_0, +1$. By Theorem 5, a necessary and sufficient condition for the existence of a function of the class E taking 0 to 0 and Z_0, \bar{Z}_0 into conjugate points, say W and \bar{W} , is that W lies in lens B described in the statement of the theorem. Denote by G the image of B under the transformation $w = (W + h')/(1 + h'W)$. We conclude that $w = f(z_0)$ must lie in G , the lens enclosing h' with end points $(Z_0 + h')/(1 + h'Z_0)$ and $(h' - Z_0)/(1 - h'Z_0)$ on C_3 and C_4 , respectively.

If $f(z)$ is real for all h , $-1 < h < +1$, we have the Carathéodory theorem [1, 53] which asserts that if $f \in E$ and if, furthermore, f is real for z real, then a point z inside lens A has its image $f(z)$ also in this lens.

Finally, we investigate the Julia inequality in the case of a reversed equilateral triangle.

THEOREM 7. *A necessary and sufficient condition for the existence of a function $f \in E$ which maps the vertices of the equilateral triangle, $r, r\omega, r\omega^2$ into the vertices of the reversed equilateral triangle $\rho, \rho\omega^2, \rho\omega$, respectively, is that $\rho \leq r^2$.*

Proof. The result is obtained by investigating the Julia condition: $D(A'_2/a'_2, A'_3/a'_3) \leq D(a'_2, a'_3)$, where

$$a'_2 = \frac{r(\omega - 1)}{1 - r^2\omega}, \quad a'_3 = \frac{r(\omega^2 - 1)}{1 - r^2\omega^2}, \quad A'_2 = \frac{\rho(\omega^2 - 1)}{1 - \rho^2\omega^2}, \quad A'_3 = \frac{\rho(\omega - 1)}{1 - \rho^2\omega}$$

and simplifying the somewhat involved expression. The computation is omitted.

In the extreme case $\rho = r^2$, the function $w = z^2$ performs the required interpolation.

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Received January 3, 1966.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 20, No. 3

November, 1967

Dallas O. Banks, <i>Lower bounds for the eigenvalues of a vibrating string whose density satisfies a Lipschitz condition</i>	393
Ralph Joseph Bean, <i>Decompositions of E^3 which yield E^3</i>	411
Robert Bruce Brown, <i>On generalized Cayley-Dickson algebras</i>	415
Richard Dowell Byrd, <i>Complete distributivity in lattice-ordered groups</i>	423
Roger Countryman, <i>On the characterization of compact Hausdorff X for which $C(X)$ is algebraically closed</i>	433
Cecil Craig, Jr. and A. J. Macintyre, <i>Inequalities for functions regular and bounded in a circle</i>	449
Takesi Isiwata, <i>Mappings and spaces</i>	455
David Lewis Outcalt, <i>Power-associative algebras in which every subalgebra is an ideal</i>	481
Sidney Charles Port, <i>Equilibrium systems of stable processes</i>	487
Jack Segal, <i>Quasi dimension type. I. Types in the real line</i>	501
Robert William Stringall, <i>Endomorphism rings of primary abelian groups</i>	535
William John Sweeney, <i>"The δ-Poincaré estimate"</i>	559
L. Tzafriri, <i>Operators commuting with Boolean algebras of projections of finite multiplicity</i>	571