

# Pacific Journal of Mathematics

## **INEQUALITIES FOR FUNCTIONS REGULAR AND BOUNDED IN A CIRCLE**

CECIL CRAIG, JR. AND A. J. MACINTYRE

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**This paper is concerned with functions  $w = f(z)$  regular and satisfying the inequality  $|f(z)| < 1$  in  $|z| < 1$ . This class of functions will be denoted  $E$ .**

**We determine conditions on  $z_1, z_2, z_3$  and  $w_1, w_2, w_3$  for**

$$w_k = f(z_k) \quad (k = 1, 2, 3)$$

**to be possible with an  $f(z)$  of  $E$ . In particular to map the vertices of the equilateral triangle  $z_k = re^{2k\pi i/3}$  into the vertices of another taken in the opposite direction  $w_k = \rho e^{-2k\pi i/3}$  we must have  $\rho \leq r^2$ . The extremal function associated with this problem is  $w = z^2$ . It seems convenient to discuss the fixed point if any of the mapping of  $|z| < 1$  into  $|w| < 1$ . We include a simple proof of the theorem of Denjoy and Wolff that if no such fixed point exists then there is a unique distinguished fixed point on  $|z| = 1$ . We give several results restricting the position of the interior or distinguished boundary fixed point in terms of the location of a zero of  $f(z)$  or the value  $f(0)$ .**

The theorem of Pick asserts that if  $f(z)$  is in  $E$  then  $D(f(z_1), f(z_2)) \leq D(z_1, z_2)$  where the nonEuclidean distance

$$D(z_1, z_2) = \frac{1}{2} \log \frac{1 + d(z_1, z_2)}{1 - d(z_1, z_2)} \quad \text{with} \quad d(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|.$$

Equality holds if and only if  $f$  sets up a Möbius transformation. It follows from Pick's theorem that there can be at most one fixed point of  $w = f(z)$  in  $|z| < 1$  unless  $f(z) \equiv z$ . It is usually sufficient when  $f$  has an interior fixed point at  $z = \alpha (\neq 0)$  to suppose  $0 < \alpha < 1$ .

Our first four theorems give information about the relative positions of zeros of  $f$ , an interior fixed point, and the value  $f(0)$ . We exclude the case where  $f(z) \equiv z$ .

**THEOREM 1.** *Let  $f \in E$  and  $f(0) \neq 0$ . Then  $f$  has no zeros in  $|z| < |f(0)|$ ; and has a zero on  $|z| = |f(0)|$  if and only if  $f$  determines a Möbius transformation.*

*Proof.* The image of  $|z| \leq |f(0)|$ , which we denote by  $C$  under the transformation  $w = (z + f(0))/(1 + \overline{f(0)}z)$  is a circular disc  $C'$  having nonEuclidean center  $f(0)$  with boundary passing through the origin. The function  $w = f(z)$  takes the closed disc  $C$  inside  $C'$  in the case  $f$

is not a Möbius transformation so that  $f(z) \neq 0$  for  $z \in C$ . If  $f$  is linear, nonEuclidean distances are preserved and  $f(0)$  is on the boundary of  $C'$ .

**THEOREM 2.** *Let  $f \in E$  and let  $z = \alpha$  be a fixed point of  $f$  with  $0 < \alpha < 1$ . Then  $f$  has no zeros inside  $|z - \alpha/(1 + \alpha^2)| = \alpha/(1 + \alpha^2)$  and has a zero on the boundary if and only if  $f$  determines a Möbius transformation.*

*Proof.* The conclusion follows directly from Pick's theorem since the circle described is the nonEuclidean circle with nonEuclidean center  $z = \alpha$ .

If  $f(0)$  is known in addition to the existence of an interior fixed point  $\alpha (\neq 0)$ , then these two results can be combined to give a larger region which is zero-free, namely the union of the two closed discs. The boundary zero of  $f$  occurs at  $z = \overline{f(0)}$  when  $f$  is a Möbius transformation.

**THEOREM 3.** *If  $f \in E$  and  $f(0) \neq 0$ , then there can be no fixed point interior to the circle  $C_1: |z| = (1 - \sqrt{1 - |f(0)|^2})/|f(0)|$ ; and a fixed point on the boundary at  $z_0 = e^{i \arg f(0)} (1 - \sqrt{1 - |f(0)|^2})/|f(0)|$  only if  $f$  determines a Möbius transformation.*

*Proof.* The nonEuclidean midpoint of the segment from 0 to  $f(0)$  is  $z_0$  (See Figure 1). A displacement of all points inside  $C_1$  by  $w = f(z)$  insures there can be no fixed point interior to  $C_1$ . The boundary case is clear.

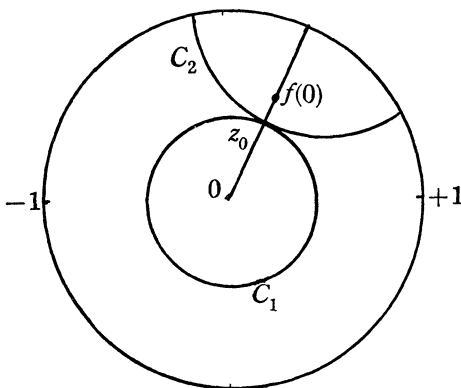


Figure 1.

If  $f$  is known to have an interior fixed point, an improvement over Theorem 3 can be made as to its location based on a knowledge of  $f(0)$ . This is indicated in:

**THEOREM 4.** *Let  $f \in E$ . If  $f$  has a fixed point  $z = \alpha (\neq 0)$ , in  $|z| < 1$ , then  $\alpha$  lies inside the circle  $C_2$  passing through  $z_0$  (Figure 1) with center at the geometric inverse of  $f(0)$ , relative to the unit circle and is on the boundary if and only if  $f$  sets up an elliptic Möbius transformation.*

*Proof.* This is a direct consequence of the inequality  $D(\alpha, 0) \geq D(\alpha, f(0))$  where the point  $z = \alpha$  is considered variable and  $f(0)$  is fixed. The assumed interior fixed point is nearer  $f(0)$  than the origin in the nonEuclidean sense, except when  $f$  is linear. This requires an investigation of the nonEuclidean perpendicular bisector of the radial segment from 0 to  $f(0)$ . Straight lines of the Poincaré model are Euclidean circles orthogonal to the unit circle. The Euclidean circle  $C_2$  passing through the point  $z_0$  and orthogonal to  $|z| = 1$  is the one described in the statement of the theorem.

Theorem 4 provides a simple proof of the Theorem of Denjoy on the fixed points of analytic transformations of the unit circle into itself [3]. It is convenient to develop the argument by formulating several variants of Theorem 4.  $f(z)$  is supposed to belong to  $E$ .

**THEOREM 4 A.** *If  $f(0) \neq 0$  and  $\arg f(0) = \theta$ , then any interior fixed point must lie in the half plane  $R(e^{i\theta}z) > 0$ .*

The half plane evidently contains the circle  $C_2$  of Theorem 4.

**THEOREM 4 B.** *The nonEuclidean bisector of the nonEuclidean segment joining  $x$  and  $f(x)$  divides the unit circle into two parts. Any interior fixed point must lie in the part containing  $f(x)$  unless the function sets up an elliptic linear transformation when the fixed point must lie on the bisector.*

This statement is equivalent to that of Theorem 4. We have only to apply Theorem 4 to  $w = Tf(T^{-1}z)$  where  $T$  is a linear transformation of  $|z| < 1$  into itself which carries  $x$  to the origin.

**THEOREM 4 C.** *If  $x$  and  $f(x)$  have the same argument, then any interior fixed point must lie on the same side as  $f(x)$  of the circle through  $x$  and orthogonal to the radius  $Ox$  and to  $|z| = 1$ .*

This follows from Theorem 4 A. We consider  $w = Tf(T^{-1}z)$  where  $T$  carries  $x$  to the origin and the diameter through  $x$  into itself.

Now consider  $z = g(w)$  the solution of  $wf(z) = z$ . From Rouché's theorem  $g(w)$  is regular and one valued for  $w$  in  $|w| < 1$ . Let  $0 < w < w' < 1$ . Apply Theorem 4 C to  $F(z) = w'f(z)$ . Let  $\alpha = g(w)$ . We know that  $F(\alpha) = w'\alpha/w$ . Any fixed point of  $F(z)$  and that is to say  $g(w')$  must lie in the smaller part of the unit circle partitioned

as in Theorem 4C. If  $g(w)$  does not tend to a fixed point of  $|w| < 1$  as  $w \rightarrow 1$  by positive values, it must converge to a point of  $|z| = 1$ . This point on  $|z| = 1$  is the Denjoy distinguished fixed point. Calling such points  $D$  fixed points it is clear that Theorem 4 applies to these as well as to interior fixed points.

We shall next be concerned with special cases of three point interpolation by  $f \in E$ . The problem first considered is that in which we require the vertices of an isosceles triangle to be mapped by  $f$  into vertices of another isosceles triangle.

**THEOREM 5.** *A necessary and sufficient condition for the existence of a function  $f \in E$  taking points  $z_0, 0, \bar{z}_0$  into  $w, 0, \bar{w}$ , respectively, is that  $w = f(z_0)$  lies in lens  $B = \{t \mid t = z_0\zeta, \zeta \in A\}$ , where  $A$  is the lens formed by the two circular arcs passing through  $-1, z_0, +1$  and  $-1, \bar{z}_0, +1$ .*

*Proof.* This follows from an inequality of G. Julia [4, 74-78] which for our problem is expressed by  $D(w/z_0, \bar{w}/\bar{z}_0) \leq D(z_0, \bar{z}_0)$ . Since  $D$  is a monotone increasing function of  $d$ , it is sufficient for our purpose to use  $d$  and we shall refer to this as the nonEuclidean distance.

Let  $\delta = |(z_0 - \bar{z}_0)/(1 - z_0^2)|$  and  $\zeta = w/z_0 = x + iy$ . Then the basic inequality becomes  $|(\zeta - \bar{\zeta})/(1 - \zeta^2)| \leq \delta$  or  $2|y|/|1 - (x + iy)^2| \leq \delta$ . On squaring and simplifying we have  $4y^2(1 - \delta^2) \leq \delta^2(1 - \{x^2 + y^2\})^2$ . After taking square roots and rearranging we obtain

$$x^2 + \left(|y| + \frac{\sqrt{1 - \delta^2}}{\delta}\right) \leq \left(\frac{1}{\delta}\right)^2.$$

If  $y \geq 0$ ,  $\zeta$  lies on or below one circular arc; for  $y < 0$ ,  $\zeta$  lies on or above the other arc, the reflection of the first in the real axis. These arcs form the boundary of a lens. To see that the boundary curves pass through  $z_0$  and  $\bar{z}_0$ , consider the case of equality  $|\zeta - \bar{\zeta}|/|1 - \zeta^2| = |z_0 - \bar{z}_0|/|1 - z_0^2|$ . This equation describes the locus of a point which is a fixed nonEuclidean distance from its conjugate, in this case the nonEuclidean distance being  $d(z_0, \bar{z}_0)$ . The lens just described is labeled  $A$  in Figure 2. To complete the proof one notes that  $w = z_0\zeta$ , for  $\zeta \in A$ , is the set of points of lens  $B$ .

A slightly more general result than Theorem 5 can be obtained. We require  $f$  to be real at a real point  $h$  as well as to take conjugate values at the conjugate pair  $z_0, \bar{z}_0$ .

**THEOREM 6.** *A necessary and sufficient condition for the existence of a function  $f \in E$  taking  $z_0, h, \bar{z}_0$  into  $w, h', \bar{w}$ , respectively, where  $h$  and  $h'$  are real numbers, is that  $w = f(z_0)$  lies in a lens*

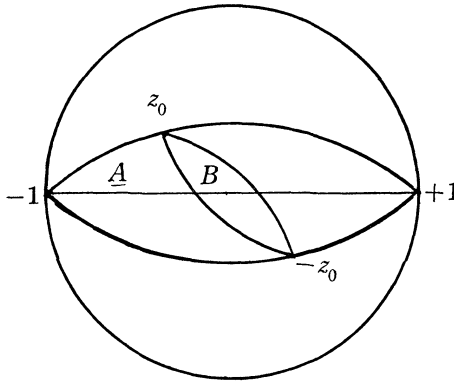


Figure 2.

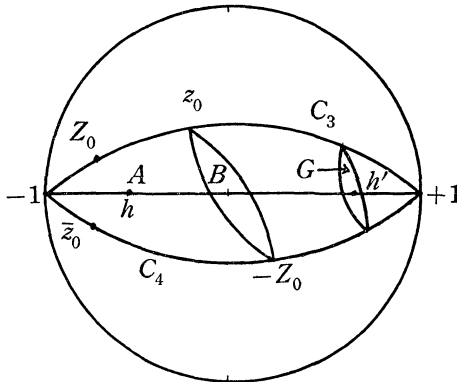


Figure 3.

$G = \{t \mid t = (W + h')/(1 + h'W), W \in B\}$ , where

$$B = \left\{ W \mid W = Z_0 \zeta, \zeta \in A, Z_0 = \frac{z_0 - h}{1 - h z_0} \right\}$$

and  $A$  is the lens described in Theorem 5.

*Proof.* The proof depends on the fact that the composition of functions in  $E$  is again in  $E$ . The transformation  $Z = (z - h)/(1 - hz)$  takes  $h$  to zero with  $z_0$  and  $\bar{z}_0$  going to conjugate points  $Z_0 = (z_0 - h)/(1 - h z_0)$  and  $\bar{Z}_0$ . Since this transformation preserves nonEuclidean distances,  $z_0$  is moved to  $Z_0$  on circular arc  $C_3$  which passes through  $-1, z_0, +1$ . By Theorem 5, a necessary and sufficient condition for the existence of a function of the class  $E$  taking 0 to 0 and  $Z_0, \bar{Z}_0$  into conjugate points, say  $W$  and  $\bar{W}$ , is that  $W$  lies in lens  $B$  described in the statement of the theorem. Denote by  $G$  the image of  $B$  under the transformation  $w = (W + h')/(1 + h'W)$ . We conclude that  $w = f(z_0)$  must lie in  $G$ , the lens enclosing  $h'$  with end points  $(Z_0 + h')/(1 + h'Z_0)$  and  $(h' - Z_0)/(1 - h'Z_0)$  on  $C_3$  and  $C_4$ , respectively.

If  $f(z)$  is real for all  $h$ ,  $-1 < h < +1$ , we have the Carathéodory theorem [1, 53] which asserts that if  $f \in E$  and if, furthermore,  $f$  is real for  $z$  real, then a point  $z$  inside lens  $A$  has its image  $f(z)$  also in this lens.

Finally, we investigate the Julia inequality in the case of a reversed equilateral triangle.

**THEOREM 7.** *A necessary and sufficient condition for the existence of a function  $f \in E$  which maps the vertices of the equilateral triangle,  $r$ ,  $r\omega$ ,  $r\omega^2$  into the vertices of the reversed equilateral triangle  $\rho$ ,  $\rho\omega^2$ ,  $\rho\omega$ , respectively, is that  $\rho \leq r^2$ .*

*Proof.* The result is obtained by investigating the Julia condition:  $D(A'_2/a'_2, A'_3/a'_3) \leq D(a'_2, a'_3)$ , where

$$a'_2 = \frac{r(\omega - 1)}{1 - r^2\omega}, \quad a'_3 = \frac{r(\omega^2 - 1)}{1 - r^2\omega^2}, \quad A'_2 = \frac{\rho(\omega^2 - 1)}{1 - \rho^2\omega^2}, \quad A'_3 = \frac{\rho(\omega - 1)}{1 - \rho^2\omega}$$

and simplifying the somewhat involved expression. The computation is omitted.

In the extreme case  $\rho = r^2$ , the function  $w = z^2$  performs the required interpolation.

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