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Let φ be a closed continuous mapping from X onto Y. It is an open problem whether the realcompactness of X implies the realcompactness of Y. Concerning this problem, in case φ is an open WZ-mapping, we discuss the structure of the image space Y under φ and give a necessary and sufficient condition that Y be realcompact. We also show that if X is locally compact, countably paracompact, normal space then the image space Y of X under a closed mapping is realcompact when X is realcompact.

The notion of realcompact space was introduced by E. Hewitt [7] under the name of Q-spaces. The importance of this notion has been recognized and investigated by many mathematicians (cf. [4, 7]). In this paper we shall discuss the relations between realcompactness and closed continuous mappings and treat also the relations between pseudocompactness and continuous mappings.

As a generalization of closed mappings[1], we have a Z-mapping. Here we shall introduce the notion of WZ-mappings as a further generalization of closed mappings. In Theorem 2.1, we shall prove that pseudocompactness of a space X is equivalent to any one of the following conditions: 1) any continuous mapping from X onto any weakly separable space is always a Z-mapping, (2) the projection: $Y \times X \rightarrow Y$ is a Z-mapping for any weakly separable space Y. We denote by $\varphi: X \rightarrow Y$ a mapping φ from X onto Y; then φ can be extended to a continuous mapping $\Phi: \beta X \rightarrow \beta Y$, called the *Stone extension* of φ, where βX and βY are the Stone Čech compactifications of X and Y resp. (In the sequel we denote always by Φ the Stone extension of φ). In § 4, we shall deal with an extension of an open mapping, and show, in Theorem 4.4, that if $\varphi: X \rightarrow Y$ is a WZ-mapping, then Φ is open if and only if φ is open. This plays an important role in § 6. We shall consider in § 5 the inverse images of realcompact space under Z-mappings. It is known that if φ is a mapping from a given space X onto a realcompact space Y, then $\Phi^{-1}(Y)$ is realcompact [4, p. 148]. In Theorem 5.3, we shall show

[1] Throughout this paper we assume that all our spaces are completely regular T_1-spaces and mappings are continuous. We use, in the sequel. the same notations as in [4]. For instance, $C(X)$ is the set of all continuous functions defined on X. A subset F of X is said to be a *zero set* if $F = \{x; f(x) = 0\}$ (briefly, $F = Z(f) = Z_X(f)$) for some $f \in C(X)$. Cl_A denotes a closure operation in a space A.

that if φ is a Z-mapping from a space X onto a realcompact space Y such that every $\varphi^{-1}(y)$, $y \in Y$, is a C^*-embedded realcompact subset of X, then X is realcompact. In particular, if X is normal and every $\varphi^{-1}(y)$, $y \in Y$, is realcompact, then realcompactness is invariant under φ^{-1}.

It is an open problem [4, p. 149] whether the realcompactness of X implies the realcompactness of Y where φ is a closed mapping from X onto Y, or even whether the realcompactness of $\Phi^{-1}(Y)$ implies the realcompactness of Y. Concerning this problem, in Theorem 6.2, we shall discuss the structure of a space Y which is the image of a realcompact space X under an open WZ-mapping. From this theorem, we shall give a necessary and sufficient condition that Y be realcompact. Moreover, from Theorem 6.2, we shall establish that if φ is an open WZ-mapping from a realcompact space X onto Y such that the boundary $\mathscr{L}\varphi^{-1}(y)$ (or $\mathscr{L}_X\varphi^{-1}(y)$) of $\varphi^{-1}(y)$, $y \in Y$, is compact, then Y is also realcompact. This is a generalization of Frolík's theorem [2] (Theorem 6.5). As a further consequence of 6.2, the realcompactness is invariant under an open WZ-mapping if a space X is any one of the following types; (1) X is locally compact, (2) X is weakly separable, (3) X is connected, (4) X is locally connected and (5) X is perfectly normal. In Theorem 7.5, we shall prove, using Frolík's theorem [3], that if X is locally compact, countably paracompact, normal space, then the image of X under a closed mapping is realcompact when X is realcompact. It seems to me that this is only one case for which realcompactness is proved to be invariant under a closed mapping without any additional condition. In the process of the proof of this theorem, we obtain that the image Y of a locally compact, realcompact, normal space under a closed mapping φ is locally compact if and only if $\mathscr{L}\varphi^{-1}(y)$ is compact for every $y \in Y$.

1. Definitions and preliminaries. $\varphi: X \to Y$ is said to be a *Z-mapping*, according to Frolík [2], if φ maps every zero set of X to a closed set of Y. Moreover we shall define a *WZ-mapping* as a further generalization of a closed mapping. φ is called a WZ-mapping if $\mathrm{cl}_{\beta X}(\varphi^{-1}(y)) = \Phi^{-1}(y)$ for every $y \in Y$. We shall say that a subset F of X has *the property* (*) if we have $\inf\{f(x); x \in F\} > 0$ for every $f \in C(X)$ which is positive on F. A subset F of X is said to be *relatively pseudocompact* if f is bounded on F for every $f \in C(X)$. A pseudocompact subset has the property (*) and a subset with the property (*) is always relatively pseudocompact, and hence every subset of a pseudocompact space is always relatively pseudocompact. We now list some properties with respect to these concepts.

1.1. *A closed mapping is always a Z-mapping.*

1.2. *A Z-mapping is always a WZ-mapping.*

Proof. Let $z \in \Phi^{-1}(y) - \mathrm{cl}_{\beta X}\varphi^{-1}(y)$; then there is $f \in C(\beta X)$ such that $f(z) = 0, f = 1$ on $\mathrm{cl}_{\beta X}\varphi^{-1}(y)$ and $0 \leqq f \leqq 1$.

$$M = X \cap \{x; f(x) \leqq 1/2,\ x \in \beta X\}$$

is a zero set of X. Since φ is a Z-mapping and $M \cap \varphi^{-1}(y) = \phi$, $\varphi(M)$ is closed and does not contain y. On the other hand, $f(z) = 0$, and hence $z \in \mathrm{cl}_{\beta X}\ M$; this implies that

$$y = \Phi(z) \in \Phi(\mathrm{cl}_{\beta X} M) \subset \mathrm{cl}_{\beta Y}\Phi(M) = \mathrm{cl}_{\beta Y}\varphi(M)\ .$$

Since $\varphi(M)$ is closed in Y, $\mathrm{cl}_{\beta Y}\varphi(M) \cap Y = \varphi(M)$, and hence, $y \in Y$ implies $y \in \varphi(M)$. This is a contradiction.

1.3. *Let $\varphi: X \rightarrow Y$ be a WZ-mapping. If either X is normal or the boundary $\mathscr{L}\varphi^{-1}(y)$, for every $y \in Y$ is compact, then φ is a closed mapping.*

Proof. Let F be a closed subset of X and let $y \notin \varphi(F)$. It is easy to see, under the assumption of 1.3, that there is $f \in C(X)$ such that $f = 0$ on $\varphi^{-1}(y)$, $f = 1$ on F and $0 \leqq f \leqq 1$. Since φ is a WZ-mapping $\mathrm{cl}_{\beta X}\varphi^{-1}(y) = \Phi^{-1}(y)$ and $g = 0$ on $\Phi^{-1}(y)$ where g is the extension of f over βX. Φ being closed, $Y - M$ is an open set containing y and $\varphi(F) \subset M$ where $M = \Phi(\{z; z \in \beta X, g(z) \geqq 1/2\})$ ($y \notin M$ is obvious). This means that $y \notin \overline{\varphi(F)}$, that is, φ is closed.

1.4. *Let F be a closed relatively pseudocompact subset of X. If either X is normal or F is a zero set of X, then F has the property* (*) (*see* 3.3 *below*).

Proof. Let f be a function of $C(X)$ and $f > 0$ on F. Now suppose that $Z(f) = E \neq \phi$. If either X is normal or $F = Z(g)$ for some $g \in C(X)$, then E and F are completely separated, i.e., there is a function $h \in C(X)$ such that $h = 1$ on E, $h = 0$ on F and $0 \leqq h \leqq 1$. Then we have $Z(|f| + h) = \phi$ which implies $k = 1/(|f| + h) \in C(X)$. If $\inf\{f(x); x \in F\} = 0$, then it is easy to see that k is not bounded on the closed relatively pseudocompact subset F. This is a contradiction.

1.5. *Every zero set of a pseudocompact space has the property* (*) (*by* 1.4).

1.6. *Suppose that* φ *is a mapping from* X *onto* Y *and every point of* Y *is* G_δ. *If a closed subset* F *of* X *has the property* (*), *then* $\varphi(F)$ *is closed.*

Proof. Let F be a closed subset of X having the property (*) and let $y \notin \varphi(F)$. Since y is a G_δ-point, there is a function $f \in C(Y)$ with $f^{-1}(0) = \{y\}$ and $0 \leqq f \leqq 1$. $h = f\varphi$ is positive on F, and hence $h > \alpha > 0$ on F because F has the property (*). If $z \in \varphi(F)$, then there is a point $x \in F$ with $\varphi(x) = z$. Thus $f(z) = f(\varphi(x)) = h(x) > \alpha$. This means that $\varphi(F) \subset f^{-1}[\alpha/2, 1]$, and hence $y \notin \overline{\varphi(F)}$, that is, $\varphi(F)$ is closed.

1.7. *If, in* 1.6, X *is pseudocompact, then* φ *is always a* Z*-mapping* (*by* 1.5 *and* 1.6).

The following theorems are known and useful in the sequel.

1.8. X *is realcompact if and only if for every point* x *in* $\beta X - X$ *there is a function* f *of* $C(\beta X)$ *such that* $f > 0$ *on* X *and* $f(x) = 0$ [4, p. 119].

1.9. X *is pseudocompact if and only if any family* $\{U_n\}$ *of open sets of* X, *with* $\overline{U_n} \cap \overline{U_m} = \phi (n \neq m)$, *is not locally finite.*

1.10. *If* $\{U_n\}$ *is a locally finite family of open sets of a space* X *with* $\bar{U}_n \cap \bar{U}_m = \phi$ $(n \neq m)$ *and* $\{a_n\}$ *is a set of given positive real numbers and* $\{x_n, x_n \in U_n\}$ *is given, then there is a function* f *of* $C(X)$ *such that* $f = 0$ *on* $X - \cup U_n$, $f(x_n) = a_n$, *and* $0 \leqq f \leqq a_n$ *on* U_n.

2. Z-mappings and pseudocompactness. A weakly separable space is a space with the first axiom of countability. The next conditions which are mutually equivalent, are known; (i) X is compact (resp. countably compact), (ii) any mapping from X onto Y is closed for any space Y (resp. any weakly separable space Y), and (iii) a projection φ: $Y \times X \rightarrow Y$ is closed for any space Y (resp. any weakly separable space Y) [5, 8, 12]. In this section, we shall establish analogous theorems about pseudocompactness by means of Z-mappings.

Suppose that X is not pseudocompact and let $\{W_n\}$ be a discrete family of open sets with $X - \cup W_n = S \neq \phi$. There are functions f and g of $C(X)$ by 1.10 such that (i) $f(x_n) = \varepsilon_n$, $\{\varepsilon_n\} \downarrow 0$ and $f = 0$ on S where x_n is a given point of W_n and (ii) $g(x_n) = n$, $g = 0$ on S and $g(x) > 0$ implies $f(x) > 0$. Then $F = \{x; g(x) \geqq 1/2\}$ is a zero set and

$$\inf \{f(x); x \in F\} = 0 .$$

This shows that F has not the property (*) and f is not a Z-mapping from X onto $f(X)$. Combining 1.5, 1.6 and the arguments above, we have the equivalences between (1), (2) and (3) in the following theorem.

THEOREM 2.1. *The following conditions are equivalent for a space X.*

(1) *X is pseudocompact.*

(2) *Every zero set of X has the property (*).*

(3) *Any mapping from X onto any space Y such that every point of Y is G_δ, is always a Z-mapping.*

(4) *The projection $\varphi: Y \times X \rightarrow Y$ is a Z-mapping for any weakly separable space Y.*

(5) *The projection $\varphi: Y \times X \rightarrow Y$ is a Z-mapping for some nondiscrete weakly separable space Y.*

Proof. (4) $\rightarrow$ (5) is obvious. We shall show (1) $\rightarrow$ (4). Suppose that there is a function $h \in C(X \times Y)$ such that $y \in \overline{\varphi(E)} - \varphi(E)$ where $E = h^{-1}(0)$. Let $\{W_n\}$ be a base of y with

$$\bar{W}_{n+1} \subset W_n \ (n = 1, 2, \cdots) .$$

Since $\varphi^{-1}(y) = \{y\} \times X$ is pseudocompact and h is positive on $\varphi^{-1}(y)$, there is a real number $\alpha > 0$ such that $h \geqq \alpha$ on $\varphi^{-1}(y)$. For each n, we choose a point y_n in $W_n \cap \varphi(E)$ (and hence $\{y_n\} \rightarrow y$) and a point (y_n, x_n) in E. If $A = \{x_n; n = 1, 2, \cdots\}$ has an accumulation point x_0, then $(y, x_0) \in E$, that is, $y = \varphi(y, x_0) \in \varphi(E)$. This is a contradiction. Thus A must be a closed discrete subset of X. Let

$$M = \{z; h(z) < \alpha/2\}$$

and $F = \{z; h(z) \leqq \alpha/2)$. We choose an open set U_n, in X, containing x_n and an open set $V_n \subset W_n$ in containing y_n Y such that

$$\bar{U}_n(\text{in } X) \cap \bar{U}_m(\text{in } X) = \phi \ (n \neq m), \ \bar{V}_n \times \bar{U}_n \subset M .$$

X being pseudocompact, there is an x_0 in $\overline{\cup \bar{U}_{n_i}} - \cup \bar{U}_{n_i}$ for some $\{n_i\}$. We have $(y, x_0) \in F$, i.e., $y = \varphi(y, x_0) \in \varphi(F)$. On the other hand, we have $\varphi^{-1}(y) \cap F = \phi$ since $F = \{z; h(z) \leqq \alpha/2\}$ and $h \geqq \alpha$ on $\varphi^{-1}(y)$. This is a contradiction.

(5) $\rightarrow$ (1) follows from the following theorem.

THEOREM 2.2. *Suppose that Y is a space in which there is a discrete subset $M = \{y_n; y = 1, 2, \cdots\}$ which has an accumulation point y_0. If the projection $\varphi: Y \times X \rightarrow Y$ is a Z-mapping, then X must be pseudocompact.*

Proof. We shall firstly show that there is a function $f \in C(Y)$ with $f(y_n) > 0$ for every $y_n \in M$ and $f(y_0) = 0$. Since Y is completely regular, there is a function $f_1 \in C(Y)$ with $f_1(y_1) = 1$, $f_1 = 0$ on some neighborhood (briefly, nbd) V_1 of y_0 and $0 \leqq f_1 \leqq 1$. Let y_{i_2} be the point such that $y_{i_2} \in M \cap Z(f_1)$ and $i_2 > m$ implies $f_1(y_m) > 0$. Then there is a function $f_2 \in C(Y)$ such that $f_2(y_{i_2}) = 1, f_2 = 0$ on some nbd V_2 of y_0, $V_2 \subset V_1$ and $0 \leqq f_2 \leqq 1$ and $Z(f_2) \subset Z(f_1)$. Let y_{i_3} be the point such that $y_{i_3} \in M \cap Z(f_2)$ and $i_3 > m$ implies $f_2(y_m) > 0$ and so on. Define $f(x) = \sum_{n=1}^{\infty}(1/2^n)f(x)$. Then $f(x)$ is continuous and $f(y_0) = 0$ and $f > 0$ on M.

If X is not pseudocompact, there is a locally finite family $\{U_n\}$ of open sets with $U_n \cap U_m = \phi$ and there is a function $h \in C(X)$ such that $h \geqq 0$ on X and $h(x_n) = 1/f(y_n)$ for some point $x_n \in U_n$ by 1.10. Define $H(y, x) = f(y)h(x)$. $H(y, x)$ is continuous on $Y \times X$ and

$$H(y_0, x) = 0$$

for every $x \in X$ and $H(y_n, x_n) = 1$ for $n = 1, 2, \cdots$. Therefore we have $\{(y_n, x_n); n = 1, 2, \cdots\} \subset H^{-1}(1)$ and hence $M \subset \varphi(H^{-1}(1))$. On the other hand, $y_0 \notin \varphi(H^{-1}(1))$. This shows that φ is not a Z-mapping.

Even if X is pseudocompact, a closed subset F of X with the property (*) is not necessarily pseudocompact. For instance, the space D constructed in [4, 5I, p. 79], which is a zero set of the pseudocompact space Ψ, is not pseudocompact.

Relating this example, we shall consider a countably compact space. If X is not countably compact, then there are a discrete closed subset $A = \{x_n; n = 1, 2, \cdots\}$ and a function $f \in C(X)$ such that

$$f(x_n) = \varepsilon_n, \{\varepsilon_n\} \downarrow 0 \text{ and } f \geqq 0 \text{ on } X.$$

It is obvious that A has not the property (*). Thus we see that X *is countably compact if and only if every closed subset of X has the property* (*).

3. Mappings and the property (*). In this section we shall consider the relations between mappings given in § 1 and the property (*), and moreover give several examples. We shall say that φ *has the property* (*) if $\varphi^{-1}(y)$ has the property (*) for every $y \in Y$.

3.1. (1) *Let $\varphi: X \to Y$ be a mapping and every $\varphi^{-1}(y), y \in Y$, be relatively pseudocompact. If φ is a Z-mapping, then φ has the property* (*).

(2) *If $\varphi: X \to Y$ is a WZ-mapping and φ has the property* (*), *then φ is a Z-mapping.*

Proof. (1). Suppose that there is a point y in Y such that $F = \varphi^{-1}(y)$ has not the property (*), that is, there exists a function $h \in C(X)$ which is positive on F, $h \geqq 0$ on X and $h(x_n) = \varepsilon_n$, $\{\varepsilon_n\} \downarrow 0$ for some sequence $\{x_n\}$ in F. We can find a family $\{W_n\}$ of open sets such that $\bar{W}_n \cap \bar{W}_m = \phi$ $(n \neq m)$, $\varepsilon_n - \rho_n \leqq h(x) \leqq \varepsilon_n + \rho_n$ on W_n where $\min\{\varepsilon_n - \varepsilon_{n+1}, \varepsilon_{n-1} - \varepsilon_n\} = 2\rho_n$, and $x_n \in W_n$. $E = h^{-1}(0)$ is not empty because $E = \phi$ implies $1/h \in C(X)$ and $1/h$ is not bounded on a relatively pseudocompact subset F. We shall show that φ is not a Z-mapping. To do this, it is sufficient to show that $y \in \overline{\varphi(E)}$ because E is a zero set and $y \notin \varphi(E)$. If $y \notin \overline{\varphi(E)}$, then there is a function $g \in C(Y)$ such that $g=1$ on $\overline{\varphi(E)}$, $g(y) = 0$ and $0 \leqq g \leqq 1$. This implies that $g\varphi \in C(X)$, $g\varphi = 1$ on E and $g\varphi = 0$ on F. The function $k = h + g\varphi$ is positive, continuous on X, and hence $1/k \in C(X)$. On the other hand, $1/k$ is not bounded on F. This contradicts the fact that F is relatively pseudocompact.

(2). Let $F = Z(f)$, $f \in C^*(X)$ and $y \notin \varphi(F)$. Since φ has the property (*), we have $\inf\{f(x); x \in \varphi^{-1}(y)\} = \alpha > 0$. Let g be an extension of f over βX; then $g \geqq \alpha$ on $\Phi^{-1}(y) = \mathrm{cl}_{\beta X}\varphi^{-1}(y)$.

$$E = \{x; x \in \beta X, g(x) \leqq \alpha/2\}$$

is compact and $y \notin \Phi(E)$. $\Phi(E)$ being compact, $V = \beta Y - \Phi(E)$ is an open subset (in βY) containing y. Thus $V \cap Y$ is an open subset (in Y) containing y and $\varphi(F) \cap (V \cap Y) \subset \Phi(E) \cap V \cap Y = \phi$. This implies that $y \notin \overline{\varphi(F)}$, that is, $\varphi(F)$ is closed which shows that φ is a Z-mapping.

From 3.1 we have

3.2. (1) *If φ is a Z-mapping from a pseudocompact space X onto Y, then φ has the property* (*).

(2) *If φ is a WZ-mapping from a countably compact space X onto Y, then φ is a Z-mapping.*

We can not replace "Z-mapping" in (1) of 3.2 by "WZ-mapping" and "Z-mapping" in (2) of 3.2 by "closed mapping" respectively, as will be seen from examples 3.4 and 3.5 below respectively.

3.3. *If F is a C^*-embedded subset of X with the property* (*), *then F is pseudocompact. In particular, in a normal space, a closed subset with the property* (*) *is always countably compact* (*see* 1.4).

Proof. If F is not pseudocompact, then there is a function $f \in C(F)$ with $1 \geqq f > 0$ and $\inf\{f(x); x \in F\} = 0$. Let g be an extension of f over X; then $g > 0$ on F and $\inf\{g(x); x \in F\} = 0$ which is a contradiction.

EXAMPLE 3.4. Let $X = W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}$,

$$Y = W(\omega_1 + 1)$$

and let $\varphi: X \to Y$ be defined by $\varphi(y, x) = y$. Every $\varphi^{-1}(y), y \in Y$, is relatively pseudocompact. Since $\beta X = W(\omega_1 + 1) \times W(\omega_0 + 1)$, we have $\Phi^{-1}(y) = \mathrm{cl}_{\beta X}\varphi^{-1}(y)$, i.e., φ is an open WZ-mapping. But φ is not a Z-mapping by (1) of 3.2 because $\varphi^{-1}(\omega_1)$ has not the property (*) and X is pseudocompact.

EXAMPLE 3.5. Let

$$X = W(\omega_1 + 1) \times W(\omega_1 + 1) - \{(\omega_1, \omega_1)\},\ Y = W(\omega_1 + 1)$$

and let $\varphi: X \to Y$ be defined by $\varphi(y, x) = y$. Every $\varphi^{-1}(y)$ is compact except $y = \omega_1$ and $\varphi^{-1}(\omega_1)$ is countably compact. Thus φ is an open Z-mapping by (2) of 3.2. But φ is not closed because

$$F = \{(y, x);\ x = \omega_1,\ y \in W(\omega_1)\}$$

is closed but $\varphi(F) = W(\omega_1)$ is not closed in Y. (We notice that X is countably compact.)

EXAMPLE 3.6. Let $X = W(\omega_1 + 1) \times W(\omega_1 + 1) - \{(y, x);\ y = \omega_1,$

$$\omega_0 < x \leqq \omega_1\},\ Y = W(\omega_1 + 1)$$

and let $\varphi: X \to Y$ be defined by $\varphi(y, x) = y$. Since

$$Z = W(\omega_1) \times W(\omega_1 + 1)$$

is pseudocompact and $\beta Z = Y \times Y, X$ is pseudocompact [9] and it is easy to see that every $\varphi^{-1}(y), y \in Y$, is compact. Thus φ is an open compact mapping but not a WZ-mapping. ($\varphi: X \to Y$ is said to be *compact* if $\varphi^{-1}(y)$ is compact for every $y \in Y$.)

4. Extensions of open mappings. For an extension of an open mapping $\varphi: X \to Y$ where both spaces X and Y are normal, the following theorem is known: if either φ is compact or φ is closed, then Φ is open ([1], in which φ is assumed to be a many-valued mapping). In this section, we shall show that if φ is a (single-valued) WZ-mapping, then we can drop the assumption of normality of both spaces; that is, φ is open if and only if Φ is open. Let $\varphi: X \to Y$ be a mapping. A function f is said to be *φ-bounded* if f is bounded on $\varphi^{-1}(y)$ for every $y \in Y$.

If $f \in C(X)$ is φ-bounded, we put

$$f^i(y) = \inf\{f(x);\ x \in \varphi^{-1}(y)\},\ f^s(y) = \sup\{f(x);\ x \in \varphi^{-1}(y)\}\ ;$$

these are real-valued functions defined on Y. The following lemma is useful.

LEMMA 4.1. ([2]). *Let $\varphi: X \to Y$ be a mapping and let $f \in C(X)$ be φ-bounded.*

(i) *If φ is open, then f^s(resp. f^i) is lower (resp. upper) semi-continuous.*

(ii) *If φ is closed, then f^s (resp. f^i) is upper (resp. lower) semi-continuous.*

(iii) *If φ is a WZ-mapping, then f^s (resp. f^i) is upper (resp. lower) semi-continuous.*

Proof. (i) and (ii) are essentially proved in [2]. (iii) is obtained in the following way: let g be the extension of f over $\Phi^{-1}(Y)$; by (ii) g^s (resp. g^i) is upper (resp. lower) semi-continuous on Y because Φ is a closed mapping. Since φ is a WZ-mapping, we have

$$g^s = f^s \text{ and } g^i = f^i .$$

This completes the proof.

If φ is an open WZ-mapping, then f^s and f^i are continuous on Y for every φ-bounded function $f \in C(X)$ by 4.1.

As applications of 4.1 we have the following 4.2 and 4.3.

4.2. *If φ is an open WZ-mapping from X onto a pseudocompact space Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for every $y \in Y$, then X is pseudocompact.*

This is a generalization of a theorem of Hanai and Okuyama [6] and our proof is simpler than theirs; that is, 4.2 follows from the facts that for any $f \in C(X)$, f is φ-bounded, and hence f^s (resp. f^i) is bounded by (iii) and continuous on Y by the note above which concludes that f is bounded on X.

4.3. *If φ is a WZ-mapping from X onto a countably compact space Y such that $\varphi^{-1}(y)$ is relatively pseudocompact for every $y \in Y$, then X is pseudocompact.*

Proof. Let f be any function of $C(X)$; then $|f|$ is φ-bounded and $|f|^s$ is upper semi-continuous by (iii). Since a space is countably compact if and only if every upper semi-continuous function is bounded above [10], we see that $|f|^s$ must be bounded above, that is, f is bounded. This means that X is pseudocompact.

THEOREM 4.4. (i) *A mapping* $\varphi: X \to Y$ *is a WZ-mapping if and only if* $\varphi(U \cap X) = \Phi(U) \cap Y$ *for every open set* U *of* βX.

(ii) *If* $\varphi: X \to Y$ *is a WZ-mapping, then* φ *is open if and only if* Φ *is open.*

Proof. (i). *Necessity.* It is sufficient to prove that $y \in \Phi(U) \cap Y$ implies $y \in \varphi(U \cap X)$. This follows from the fact that

$$\varphi^{-1}(y) \cap (U \cap X) \neq \phi$$

if and only if $\Phi^{-1}(y) \cap U \neq \phi$ for every open set U of βX since φ is a WZ-mapping.

Sufficiency. If $x \in \Phi^{-1}(y) - \mathrm{cl}_{\beta X}\varphi^{-1}(y)$, then there is an open set U (in βX) containing x which is disjoint from $\mathrm{cl}_{\beta X}\varphi^{-1}(y)$. This means that $y \notin \varphi(U \cap X)$, which contradicts $y \in \Phi(U)$.

(ii). It is sufficient, by (i), to show that the openness of φ implies the openness of Φ. Let x^* be any point in βX and let U be an open set of βX containing x^*. There exists a function $f \in C(\beta X)$ such that $0 \leqq f \leqq 1, f(x^*) = 1, f = 0$ on $\beta X - U$ and $\mathrm{cl}_{\beta X} V \subset U$ where

$$V = \{x; f(x) > 0\} .$$

We have, by 4.1, $(f \mid X)^s \in C(Y)$. Let us denote by g the extension of $(f \mid X)^s$ over βY. Then $g(\Phi(x^*)) = 1$ and $W = \{y; g(y) > 1/2\}$ is open in βY. We shall prove that $W \subset \Phi(\mathrm{cl}_{\beta X} V)$. Suppose that there is a point z in W such that $\Phi^{-1}(z) \cap \Phi^{-1}\Phi(\mathrm{cl}_{\beta X} V) = \phi$. Then $f = 0$ on $\Phi^{-1}(S)$ where S is an open subset, contained in W, containing z with $S \cap \Phi(\mathrm{cl}_{\beta X} V) = \phi$. This implies that $g \mid Y = 0$ on S which is impossible.

This theorem will be used in § 6.

5. Inverse images of realcompact spaces. Let α be a collection of coverings of X. A centred family $\mathscr{M}$ of subsets of X (i.e., with the finite intersection property) is said to be *α-Cauchy* if for every $\mathfrak{A} \in \alpha$, there exist $A \in \mathfrak{A}$ and $M \in \mathscr{M}$ with $M \subset A$. We shall say that α is *complete* if

$$\cap \overline{\mathscr{M}} \neq \phi$$

for every α-Cauchy $\mathscr{M}$, according to Frolík [3]. In the sequel, we consider only countable coverings consisting of cozero-sets where a set is said to be a *cozero-set* if it is the complement of a zero set. We denote by α_c the collection of all such coverings and moreover by α_{pc}(resp. α_{1c} and α_{sc}) the subcollection of α_c with the point-finite property (resp. with the locally finite property and with the star-finite property). If α is a collection of countable coverings of X, then define $\overline{\mathfrak{A}}^\beta = \cup\{\mathrm{cl}_{\beta X}A; A \in \mathfrak{A}\}$ for every $\mathfrak{A} \in \alpha$. $\overline{\mathfrak{A}}^\beta$ is σ-compact and hence

$Z = \cap\{\mathfrak{A}^{\beta}; \mathfrak{A} \in \alpha\}$ is realcompact and $X \subset \sim X \subset Z \subset \beta X$ where υX denotes the Hewitt's realcompactification of X.

LEMMA 5.1. *Let $\mathscr{M}$ be a centred maximal family of zero sets. Then $\mathscr{M}$ is α-Cauchy if and only if $\mathscr{M}$ has the countable intersection property where α is any one of $\alpha_c, \alpha_{pc}, \alpha_{1c}$ and α_{sc}.*

Proof. Necessity. Suppose that there is $\{Z_n\}$ in $\mathscr{M}$ with

$$\cap Z_n = \phi$$

where $Z_n = Z(f_n), 0 \leqq f_n \leqq 1$ and $f_n \in C(X)$. Then $f = \sum(f_n/2^n)$ is a positive continuous function on X.

$$A_n = \{x; 1/(n+2) < f(x) < 1/n\}$$

is a cozero-set because $A_n = X - Z(g_n)$ where $g_n = (-|f - a| + a) \vee 0$ and $a = (1/(n+2) + 1/n)/2$. It is easy to see that $\mathfrak{A} = \{A_n\} \in \alpha_{sc}$. If there is $Z \in \mathscr{M}$ with $Z \subset A_n$ for some n, then

$$B = Z \cap Z_1 \cap \cdots \cap Z_{n+2} \neq \phi$$

and we have $1/(n+2) < f < 1/n$ on B. On the other hand,

$$f < 1/(n+2)$$

on B by the method of construction of f. Thus $\mathscr{M}$ is not α_{sc}-Cauchy.

Sufficiency. It is sufficient to show that if $\mathfrak{M}$ is not α_c-Cauchy, then $\mathscr{M}$ has not the countable intersection property. Since $\mathscr{M}$ is not α_c-Cauchy, there exists

$$\mathfrak{A} = \{A_n; A_n = Z_n^c, Z_n = Z(f_n), f_n \in C(X)\} \in \alpha_c$$

such that $M \not\subset A_n$ for every n and every $M \in \mathscr{M}$. Hence $M \cap Z_n \neq \phi$ for every $M \in \mathscr{M}$. $\mathscr{M}$ being maximal, $Z_n \in \mathscr{M}$. Since $\{Z_n^c\}$ is a covering of X, we have $\cap Z_n = \phi$, and hence $\mathscr{M}$ has not the countable intersection property.

LEMMA 5.2. *The following statements are equivalent.*

(1) *X is realcompact.*

(2) *A centred maximal family of zero sets with the countable intersection property has the total nonempty intersection.*

(3) *α is complete where α is any one of $\alpha_c, \alpha_{pc}, \alpha_{1c}$ and α_{sc}.*

Proof. (1) $\leftrightarrow$ (2) is already proved in [4].

(3) $\rightarrow$ (1). If $p \in \upsilon X - X$, then $\mathscr{M} = \{Z; p \in \mathrm{cl}_{\beta X} Z, Z$ is a zero set of $X\}$ is a maximal centred family with the countable intersection

property, and hence by 5.1, $\mathscr{M}$ is α_c-Cauchy. Since α_c is complete, $\cap \mathscr{M} \neq \phi$ and it is obvious that $\cap\{\mathrm{cl}_{\beta x} Z : Z \in \mathscr{M}\} = \{p\}$. This is a contradiction, that is, $\upsilon X = X$.

(1) $\rightarrow$ (3). It is sufficient to prove that the realcompactness implies the completeness of α_{sc}. Let α_N be the family of all countable normal open coverings; then α_N is complete since X is realcompact. On the other hand, α_{sc}-Cauchy family is α_N-Cauchy family. Therefore we see α_{sc} is complete.

THEOREM 5.3. *Let $\varphi: X \rightarrow Y$ be a Z-mapping and let every $\varphi^{-1}(y)$, $y \in Y$, be a C^*-embedded realcompact subset of X. If Y is realcompact, then so is also X.*

Proof. Let $\mathscr{M}$ be a maximal centred α_c-Cauchy family consisting of zero sets of X; then $\mathscr{M}$ has the countable intersection property by 5.1. Thus by 5.2 it is sufficient to show that $\mathscr{M}$ has the total nonempty intersection. Since $\varphi(\mathscr{M})$ is α_c-Cauchy (in Y) and Y is realcompact, we have $y \in \cap \varphi(\mathscr{M})$ for some point y by 5.2. φ being a Z-mapping, $\varphi(M) = \overline{\varphi(M)}$ for every $M \in \mathscr{M}$. Since $M, N \in \mathscr{M}$ implies $M \cap N \in \mathscr{M}$, $\mathscr{M} \cap \varphi^{-1}(y)$ has the finite intersection property on $\varphi^{-1}(y)$. Let $\mathfrak{A} = \{\varphi^{-1}(y) - Z(g_n); n = 1, 2, \cdots\}$ be a covering of $\varphi^{-1}(y)$ where $g_n \in C(\varphi^{-1}(y))$ and g_n is bounded. Without loss of generality we can assume that $0 \leqq g_n \leqq 1$ for each n. Let f_n be an extension of g_n over X and define $f = \sum (f_n/2^n)$. f is continuous and $Z(f) \cap \varphi^{-1}(y) = \phi$. Y being completely regular and φ being a Z-mapping, there is $h \in C(Y)$ with $0 \leqq h \leqq 1$, $h(\varphi Z(f)) = 1$ and $h(y) = 0$.

$$\{X - Z(h\varphi), X - Z(f_n); n = 1, 2, \cdots\}$$

is a covering of X. We shall show that $M \not\subset X - Z(h\varphi)$ for every $M \in \mathscr{M}$. Suppose that there is a set $M \in \mathscr{M}$ such that

$$M \subset X - Z(h\varphi) .$$

Since $\varphi^{-1}(y) \subset Z(h\varphi)$, we have $M \cap \varphi^{-1}(y) = \phi$, but this contradicts the fact that $M \cap \varphi^{-1}(y) \neq \phi$ for every $M \in \mathscr{M}$. Thus there are $M \in \mathscr{M}$ and n with $M \subset X - Z(f_n)$, that is, $\mathscr{M} \cap \varphi^{-1}(y)$ is α_c-Cauchy (on $\varphi^{-1}(y)$). Since $\varphi^{-1}(y)$ is realcompact, we have $\cap(\mathscr{M} \cap \varphi^{-1}(y)) \neq \phi$. This means $\cap \mathscr{M} \neq \phi$. Therefore X is realcompact.

THEOREM 5.4. *If φ is a closed mapping from a normal space X to a realcompact space Y such that every $\varphi^{-1}(y)$, $y \in Y$, is realcompact, then X is also realcompact.*

6. Open WZ-mappings and realcompactness. A point p is said to be a *P-point of* X if every continuous function defined on X is constant on some nbd of p. A space X is called a *P-space* if every point of X is a P-point of X.

In the following, let $\varphi\colon X \to Y$ be an open WZ-mapping, and we divide both spaces X and Y into classes in the following way: $X_d = \{x; \varphi(x)$ is isolated and $\varphi^{-1}\varphi(x)$ is not compact$\}$, $X_{cd} = \{x; \varphi(x)$ is isolated and $\varphi^{-1}\varphi(x)$ is compact$\}$, $X_e = \{x; x \notin X_d \cup X_{cd}$ and $\varphi^{-1}\varphi(x)$ is not compact$\}$,

$$X_{ce} = X - X_d - X_{cd} - X_e,\ Y_d = \varphi(X_d),$$
$$Y_{cd} = \varphi(X_{cd}),\ Y_e = \varphi(X_e) \text{ and } Y_{ce} = \varphi(X_{ce}) .$$

LEMMA 6.1. *If $\varphi\colon X \to Y$ is an open WZ-mapping, $y^* \in Y_e$ and if there is a function $f \in C(\beta X)$ such that $0 \leqq f \leqq 1, f > 0$ on X and $f(x^*) = 0$ for some $x^* \in \Phi^{-1}(y^*) - \varphi^{-1}(y^*)$, then $Z_{\beta X}(f^i\Phi)$ is a neighborhood (in βX) of $\Phi^{-1}(y^*)$, equivalently, $Z_{\beta Y}(f^i)$ is a neighborhood (in βY) of y^*. (We notice that Φ is open by* 4.4)

Proof. Suppose that $Z_{\beta Y}(f^i)$ is not a nbd of y^*, i.e., $Z_Y(f^i)$ is not a nbd of y^*. Let us put $h = f^i \mid Y$, $\alpha_{2n} = 1/2n - 1/(2n+1)$ and

$$\begin{aligned} a_n &= 1/2n - (4/7)\cdot\alpha_{2n}, \\ b_n &= 1/2n + (4/7)\cdot\alpha_{2n-1} \\ c_n &= 1/(2n+1) - (4/7)\cdot\alpha_{2n+1}, \\ d_n &= 1/(2n+1) + (4/7)\cdot\alpha_{2n} \\ F_n &= \varphi^{-1}h^{-1}[a_n, b_n], \\ E_n &= \varphi^{-1}h^{-1}[c_n, d_n] . \end{aligned}$$

It is easy to see that either $\mathrm{cl}_{\beta X}(\cup F_n)$ or $\mathrm{cl}_{\beta X}(\cup E_n)$ contains x^*, say $\mathrm{cl}_{\beta X}(\cup F_n) \ni x^*$. Let us put $q_n = (f_n - b_n) \vee 0$ and

$$k_n = |\,h\varphi - \beta_n\,| \vee \{b_n - \beta_n\} - \{b_n - \beta_n\}$$

where $\beta_n = (a_n + b_n)/2$; then $q_n \in C(\beta X)$, $k_n \in C(X)$, $A_n = \{x; x \in \beta X, f(x) \leqq b_n\} = Z_{\beta X}(q_n)$, $F_n = Z_X(k_n)$ and $\{G_n; n = 1, 2, \cdots\}$ is locally finite family of zero sets of X where $G_n = Z_X(q_n + k_n) = F_n \cap A_n$. We can assume that every G_n is not empty.

Next we shall prove that $\cup G_n$ is a zero set. If we put

$$t_n = 1/2n - (5/7)\cdot\alpha_{2n},\ s_n = 1/2n + (5/7)\cdot\alpha_{2n-1}$$

and $B_n = \{x; x \in \beta X, f(x) < s_n\}$, then $U_n = \varphi^{-1}h^{-1}(t_n, s_n)$ is an open set containing F_n and $W_n = U_n \cap B_n$ is also an open set such that $G_n \subset W_n$ and $\overline{W}_n \subset \varphi^{-1}h^{-1}[t_n, s_n]$. Since $\overline{W}_n \cap \overline{W}_m = \phi$ and $x \in \overline{\cup \overline{W}_n} - \cup \overline{W}_n$ implies $f(x) = 0$, $\{W_n\}$ is a discrete collection of open sets of X because $f > 0$ on X. If $x \notin B_n$, then $f(x) \geqq s_n$, $k_n(x) \geqq 0$, and hence

$$k_n(x) + q_n(x) \geqq q_n(x) > s_n - b_n a_n - t_n = p_n > 0 .$$

If $x \notin U_n$, then $|h\varphi(x) - \beta_n| > \beta_n - t_n$, $q_n(x) \geqq 0$, and hence

$$k_n(x) + q_n(x) \geqq k_n(x) > \beta_n - t_n - b_n + \beta_n = a_n - t_n = p_n > 0 .$$

Let us put $g_n(x) = \{(k_n(x) + q_n(x)) \wedge p_n\} \times (1/p_n)$. Then

$$g_n = 1 \text{ on } X - W_n \text{ and } x \in G_n$$

if and only if $g_n(x) = 0$. Define

$$g(x) = \begin{cases} 1 & \text{for } x \in X - \cup W_n \\ g_n(x) & \text{for } x \in W_n - G_n \\ 0 & \text{for } x \in \cup G_n . \end{cases}$$

Since $\{W_n\}$ is a discrete collection, $g(x)$ is continuous and $Z(g) = \cup G_n$, that is, $\cup G_n$ is a zero set.

Since $Z(g) \cap Z(h\varphi) = \phi$, we have $\mathrm{cl}_{\beta X} Z(g) \cap \mathrm{cl}_{\beta X} Z(h\varphi) = \phi$, and hence $y^* \notin \Phi(Z(g))$ because $\mathrm{cl}_{\beta X} Z(h\varphi) \supset \Phi^{-1}(y^*)$(notice; φ is a WZ-mapping).

Replacing a_n, b_n, t_n and s_n by $a'_n = 1/2n - (5/7) \cdot \alpha_{2n}$, $b'_n = 1/2n + (5/7) \cdot \alpha_{2n-1}$, $t'_n = 1/2n - (6/7) \cdot \alpha_{2n}$ and $s'_n = 1/2n + (6/7) \cdot \alpha_{2n-1}$ respectively, we can define and construct F'_n, q'_n, β'_n, k'_n, A'_n, G'_n, p'_n, g'_n and g' using methods similar to definitions and constructions of F_n, q_n, β_n, k_n, A_n, G_n, p_n, g_n and g respectively in the arguments above. Then

$$G_n \subset G'_n,\ Z(g) \subset Z(g'),\ Z(g') \cap Z(h\varphi) = \phi$$

and $y^* \notin \Phi(Z(g'))$. Thus there exists a nbd W(in Y) of y^* with

$$W \cap \Phi(Z(g')) = \phi .$$

On the other hand, $x^* \in \mathrm{cl}_{\beta X}(\cup F_n)$ and $y^* \in Y$ implies $y^* \in \overline{\cup \varphi(F_n)}$, and hence there is a point y in $\varphi(F_m) \cap W$ for some m, that is

$$a_m \leqq h(y) \leqq b_m .$$

This shows that there exists a point x of $\varphi^{-1}(y)$ with $x \in A'_m$ and $x \in F'_m$. Since $G'_m = A'_m \cap F'_m$, $y \in \varphi(G'_m)$. This contradicts $W \cap \Phi(Z(g')) = \phi$.

The following theorem indicates the structure of the image of a realcompact space under an open WZ-mapping.

THEOREM 6.2. *Let φ be an open WZ-mapping from a realcompact space X onto Y.*

(i) *Every point $y \in Y_e$ is a nonisolated P-point of Y, and hence $Y_e \cup Y_d$ is an open P-subspace of Y and $Y_{ce} \cup Y_{cd}$ is closed in Y.*

(We shall prove in 6.5 that $Y_e = \phi$ implies the realcompactness of Y).

(ii) *If Y is not realcompact, then every point y^* of $\upsilon Y = Y$ is a P-point of βY and $Y_{ce} \cup Y_{cd}$ is closed in υY.*

Proof. (i). Let $y \in Y_e$ and $h \in C(\beta Y)$ with $h(y) = 0$ and let

$$x^* \in \Phi^{-1}(y) - \varphi^{-1}(y) .$$

X being realcompact, there is a function $f \in C(\beta X)$ such that

$$0 \leqq f \leqq 1, f(x^*) = 0$$

and $f > 0$ on X. $k = f + h\Phi$ is continuous and $k > 0$ on X and

$$k(x^*) = 0 .$$

By 6.1, $Z(k^i)$ is a nbd (in βY) of y. On the other hand $k^i \geqq h$ implies $Z(k^i) \subset Z(h)$. This shows that h vanishes on some nbd of y, i.e., y is a P-point of Y. Thus $Y_e \cup Y_d$ becomes to be a P-space. Since $k^i(y) > 0$ for every $y \in Y_{ce} \cup Y_{cd}$, $Y_e \cup Y_d$ is open in Y and hence $Y_{ce} \cup Y_{cd}$ is closed in Y.

(ii). Let $y^* \in \upsilon Y - Y, x^* \in \Phi^{-1}(y^*)$ and let f be a function of $C(\beta X)$ with $0 \leqq f \leqq 1, f(x^*) = 0, f > 0$ on X. Let us put $X_0 = \Phi^{-1}(Y)$. If $Z_{\beta X}(f) \cap X_0 = \phi$, then $Z_{\beta Y}(f^i) \cap Y = \phi$ since every $\Phi^{-1}(y), y \in Y$, is compact and $f > 0$ on X_0, and hence $f^i > 0$ on Y and $f^i(y^*) = 0$. Thus we have $1/f^i \in C(Y)$ and $1/f^i$ can not be continuously extended over y^*. But this is impossible since $y^* \in \upsilon Y - Y$. Thus we have $Z_{\beta X}(f) \cap X_0 \neq \phi$ which implies $Z_Y(f^i) \neq \phi$. For every $y \in Y_{ce} \cup Y_{cd}$, $f > \alpha(y)$ on $\varphi^{-1}(y)$ because $\varphi^{-1}(y)$ is compact where $\alpha(y)$ is some real number. $Z_{\beta Y}(f^i) \cap Y$ is an open-closed subset of $Y(\subset Y_e \cup Y_d)$ by (i) and $\mathrm{cl}_{\beta X}(Z(f^i) \cap Y)(\subset \mathrm{cl}_{\beta Y} Z(f^i) = Z(f^i))$ is also open-closed in βY. This shows that $y^* \in \upsilon(Z(f^i) \cap Y)$ because

$$\upsilon Y = \upsilon(Z(f^i) \cap Y) \cup \upsilon(Y - Z(f^i))$$

and $\upsilon(Z(f^i) \cap Y) \cap \upsilon(Y - Z(f^i)) = \phi$ (we notice $Z(f^i) = Z_{\beta Y}(f^i)$). Since $Z(f^i) \cap Y$ is a P-space, so is also $\upsilon(Z(f^i) \cap Y)$ and every point of $\upsilon(Z(f^i) \cap Y)$ is a P-point of $\upsilon(Z(f^i) \cap Y)$ and hence of βY [4, p. 211].

From the argument above, every point $y^* \in \upsilon Y - Y$ has a nbd which is disjoint from $Y_{ce} \cup Y_{cd}$, and by (i) every point of $Y_e \cup Y_d$ has also a nbd which is disjoint from $Y_{ce} \cup Y_{cd}$. Thus $Y_{ce} \cup Y_{cd}$ is closed in υY.

If $\beta Y - Y$ contains a P-point p of βY, then it is known that every function $f \in C(Y)$ can be continuously extended over p, and hence, Y is not realcompact. The converse is not necessarily true.

Such an example is given by the space in Example 3.4, that is, $Y = W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}$ is not realcompact but $\beta Y - Y$ consists of only one point (ω_1, ω_0) which is not a P-point of βY.

But if Y is the image of a realcompact space X under an open WZ-mapping, then Theorem 6.2 concludes the following: the fact that Y is not realcompact implies that $\beta Y - Y$ contains a P-point of βY. Thus the equivalence of (1) and (2) in the following Theorem 6.3 is obtained.

Let $y^* \in \beta Y - Y$. We denote by $0(y^*)$ the set of all functions of $C(X)$ such that $\mathrm{cl}_{\beta X} Z_X(f)$ is a nbd of $\Phi^{-1}(y^*)$, and

$$Z(0(y^*)) = \{Z_X(f); f \in 0(y^*)\}.$$

$0(y^*)$ is a Z-ideal of $C(X)$.

THEOREM 6.3. *Let φ be an open WZ-mapping from a realcompact space X onto Y; then the following statements are equivalent.*

(1) *Y is realcompact.*

(2) *There is no P-point of βY in $\beta Y - Y$.*

(3) *$Z(0(y^*))$ is not closed under countable intersection for every $y^* \in \beta Y - Y$.*

(4) *There is a function $g \in C(\beta X)$ such that $\Phi^{-1}(y^*) \subset Z_{\beta x}(g)$ but $Z_{\beta X}(g)$ is not a nbd of $\Phi^{-1}(y^*)$ for every $y^* \in \beta Y - Y$.*

Proof. (2)$\to$(3). Suppose that there is a point y^* such that $Z(0(y^*))$ is closed under countable intersection. Let g be any function of $C(\beta Y)$ with $0 \leqq g \leqq 1$ and $g(y^*) = 0$; then it is sufficient to show that $Z_{\beta Y}(g)$ is a nbd of y^*, i.e., y^* is a P-point of βY. Put $g = (g_n \vee 1/n) - 1/n$ and $f_n = g_n \mid Y$. It is obvious that $\mathrm{cl}_{\beta Y} Z_Y(f_n)$ is a nbd of y^*, $f_n\varphi \in C(X)$ and $\varphi^{-1}Z_Y(f_n) = Z_X(f_n\varphi)$. If $\mathrm{cl}_{\beta X} Z_X(f_n)$ is not a nbd of $\Phi^{-1}(y^*)$, then $Z_X(f_n\varphi)$ does not contain $X \cap U$ for any nbd U of $\Phi^{-1}(y^*)$. Since φ is open and $\varphi(Z_X(f_n\varphi)) = \varphi\varphi^{-1}Z_Y(f_n) = Z_y(f_n)$, $\varphi(X \cap U)$ is open and $\varphi(X \cap U)$ is not contained in $Z_Y(f_n)$. This contradicts the fact that $\mathrm{cl}_{\beta Y} Z_Y(f_n)$ is a nbd of y^*. Therefore $\mathrm{cl}_{\beta X} Z_X(f_n\varphi)$ is a nbd of $\Phi^{-1}(y^*)$. Since $Z_X(f_n\varphi) \in Z(0(y^*))$ and $Z(0(y^*))$ is closed under countable intersection, there is a function $k \in 0(y^*)$ with $\cap Z_X(f_n\varphi) = Z_X(k)$. Since $k \in 0(y^*)$, $\mathrm{cl}_{\beta X} Z(k)$ is a nbd of $\Phi^{-1}(y^*)$ and $\Phi(\mathrm{cl}_{\beta X} Z_X(k))$ is a nbd of y^* because Φ is open by 4.4. On the other hand, $x \in Z_X(k)$ implies $(f_n\varphi)(x) = 0$ for every n, and hence we have $\varphi(x) \in Z_Y(g \mid Y)$, i.e., $\varphi(Z_X(k)) \subset Z_Y(g \mid Y)$. We have

$$\Phi(\mathrm{cl}_{\beta X} Z_X(k)) \subset \mathrm{cl}_{\beta Y} \Phi(Z_X(k)) = \mathrm{cl}_{\beta Y}(\varphi Z_X(k)) \subset \mathrm{cl}_{\beta Y} Z_Y(g \mid Y) \subset Z_{\beta Y}(g) .$$

This shows that $Z_{\beta Y}(g)$ is a nbd of y^*.

(3)$\to$(4). Since $Z(0(y^*))$ is not closed under countable intersection, there is a function $f_n \in 0(y^*)(n = 1, 2, \cdots)$ and $\mathrm{cl}_{\beta X}(\cap Z_X(f_n))$ is not a nbd of $\Phi^{-1}(y^*)$. Let $f = \sum (1/2^n)(|f_n|/(1 + |f_n|))$. If

$$z^* \in \Phi^{-1}(y^*) - \mathrm{cl}_{\beta X} Z_X(f) ,$$

there is a compact nbd F of z^* such that $F \cap \mathrm{cl}_{\beta X} Z_X(f) = \phi$. Since X is dense in βX, we have that $F \cap X \neq \phi$ and $f > \alpha$ on $F \cap X$ for some $\alpha > 0$. This means that $f_n > \alpha_n$ on $F \cap X$ for some $\alpha_n > 0$, i.e., $\mathrm{cl}_{\beta X} Z_X(f_n)$ does not contain z^*. This is a contradiction. Thus

$$\Phi^{-1}(y^*) \subset \mathrm{cl}_{\beta X} Z_X(f) .$$

Let g be an extension of f over βX, then it is obvious that

$$\Phi^{-1}(y^*) \subset Z(g) .$$

On the other hand, $Z(g)$ is not a nbd of y^* because $\mathrm{cl}_{\beta X} Z_X(f)$ is not a nbd of y^*. Therefore the function g is a desired function in (4).

(4) $\rightarrow$ (2). Let y^* be any point in $\beta Y - Y$ and let g be a function described in the assumption (4). Without loss of generality we can assume that $g \geqq 0$. Since Φ is open and closed by 4.4 and

$$\Phi^{-1}(y^*) \subset Z_{\beta X}(g) ,$$

g^s is continuous on βY by 4.1 and $g^s(y^*) = 0$. Since $Z_{\beta X}(g)$ is not a nbd of $\Phi^{-1}(y^*)$, $\Phi(\beta X - Z_{\beta X}(g))$ is open and does not contain y^* but $\mathrm{cl}_{\beta Y}\Phi(\beta X - Z_{\beta X}(g))$ contains y^*. By the method of the construction of g^s, we see that $g^s > 0$ on $\Phi(\beta X - Z_{\beta X}(g))$ and hence

$$Z_{\beta Y}(g^s) \subset \beta Y - \Phi(\beta X - Z_{\beta X}(g)) .$$

Thus $Z_{\beta Y}(g^s)$ is not a nbd of y^*, that is, y^* is not a P-point of βY.

COROLLARY 6.4. *If φ is an open WZ-mapping from a realcompact space X onto a pseudocompact space Y, then Y must be compact.*

Proof. If Y is not compact, then $\beta Y = \upsilon Y \neq Y$ and $Y_{ce} \cup Y_{cd}$ is compact by 6.2. $Z = \beta Y - Y_{ce} - Y_{cd}$ is an open locally compact subspace of βY. Since every point z of $Z - Y$ is a P-point of βY by 6.2, z has the compact nbd which is a P-space. On the other hand, a countably compact P-space is a finite set, and hence, z must be isolated. This is a contradiction, since $z \in \beta Y - Y$.

Frolík [2] has proved the following

THEOREM (F_1). *The realcompactness is invariant under an open perfect mapping where $\varphi: X \rightarrow Y$ is said to be perfect if φ is closed and compact.*

The following theorem is a generalization of Theorem (F_1).

THEOREM 6.5. *If φ is an open closed mapping from a realcompact space X onto a space Y such that $\mathscr{L}\varphi^{-1}(y)$ is compact for every $y \in Y$ (equivalently $Y_e = \phi$), then Y is also realcompact.*

Proof. Since every $\mathscr{L}\varphi^{-1}(y)$ is compact, we have

$$Y = Y_{ce} \cup Y_{cd} \cup Y_d \text{ and } Y_{ce} \cup Y_{cd}$$

is closed in υY by 6.2. If $y^* \in \upsilon Y - Y$, then y^* is a P-point of βY by 6.2, and hence there exists an open-closed nbd W(in βY) of y^* with $V = W \cap Y \subset Y_d$. Let x_α be any point in $\varphi^{-1}(y_\alpha)$, $y_\alpha \in V$, and $A = \{x_\alpha\}$. A is a discrete closed subset of X. Since A is a closed subset of a realcompact space, A is realcompact. V is homeomorphic with A, and hence V is realcompact. V being open-closed, we have

$$y^* \in \upsilon V \subset W .$$

This contradicts $V = \upsilon V$. Thus Y must be realcompact.

REMARK. It seems to me that Theorem 6.5 is not obtained directly from Theorem (F_1) in the usual method below.

Let φ be a mapping in 6.5. For $y \in Y_{ce}$ (notice $Y_e = \phi$),

$$\varphi^{-1}(y) = \mathscr{L}\varphi^{-1}(y)$$

and it is compact. For $y \in Y_{cd} \cup Y_d$, $\varphi^{-1}(y)$ is open-closed. We consider a subset $X_0 = X_{ce} \cup X_{cd} \cup \{z;\, z$ is the point of $\varphi^{-1}(y),\ y \in Y_d\}$. Then X_0 is a closed subset of X, and hence, it is realcompact. Let φ_0 be a mapping from X_0 onto Y defined by $\varphi_0(x) = \varphi(x)$. It is obvious that φ_0 is a perfect mapping, but, from such a construction φ_0 is not in general necessarily open (if in this case, φ_0 is open, then 6.5 is an immediate consequence of Theorem (F_1)). For instance, let $N = \{t_n\}$ be the set of all natural numbers, $A_n = N$, $B_n = \beta A_n$ and let $C_n = B_n - A_n$ $(n = 1, 2, \cdots)$. We denote by M the topological sum of A_n. Then $B_n \subset \beta M$ and B_n is open in BM. Let us put

$$Z_1 = Z_2 = \beta M$$

and we define a mapping ψ_i from Z_i onto $Y = \beta N$ by the Stone extension of the mapping λ_i from M onto N with $\lambda_i(A_n) = t_n$ $(i = 1, 2)$. Since λ_i is open-closed, ψ_i is also open-closed by 4.4. Let X be the topological sum of $Z_1 - \cup C_n$ and Z_2 and define a mapping φ from X onto Y by $\varphi \mid (Z_1 - \cup C_n) = \psi_1 \mid (Z_1 - \cup C_n)$ and $\varphi \mid Z_2 = \psi_2$. We shall prove the openness of φ. Since $\varphi' = \varphi \mid (Z_1 - \cup C_n)$ is a WZ-mapping from $Z_1 - \cup C_n$ onto Y and ψ_1 is an extension mapping of φ' from $\beta(Z_1 - \cup C_n) = Z_1$ onto Y, we have by 4.4 that φ' is open. Thus it is easy to see that φ is open. Next we shall prove the closedness of φ. To do this, it is sufficient to show that $\varphi|(Z_1 - \cup C_n)$ is closed. Let F be a closed subset of $Z_1 - \cup C_n$. Since B_n is open in Z_1,

$$\mathrm{cl}_{Z_1} F \cap B_n \neq \phi$$

implies $F \cap A_n \neq \phi$. Thus we have $\psi_1(\mathrm{cl}_{Z_1} F) = \varphi(F)$, i.e., φ is closed.

Let a_n be the point of $A_n \subset Z_1$ $(n = 1, 2, \cdots)$ and let $A = \{a_n\}$ and $X_0 = (Z_1 - \cup B_n) \cup \mathrm{cl}_{Z_1} A \cup (Z_2 - \cup B_n)$ and $\varphi_0 = \varphi \mid X_0$. Since X_0 is closed in X, φ_0 is a mapping considered in the begining of this remark. $U = X_0 - \mathrm{cl}_{Z_1} A$ is open in X_0 but $\varphi_0(U)$ is contained in $Y - N$, and hence, $\varphi_0(U)$ is not open. This shows that φ_0 is not an open mapping.

By 6.5, it is proved that if $\varphi: X \rightarrow Y$ is an open WZ-mapping and if some condition imposed on X implies $Y_e = \phi$, then Y is realcompact when X is realcompact. There exist many examples of such conditions. For instance, we have the following theorem.

THEOREM 6.6. *Let φ be an open WZ-mapping from a realcompact space X onto Y. If X is any one of the following spaces, then Y is realcompact.*

(1) *X is weakly separable.*
(2) *X is locally compact.*
(3) *X is connected.*
(4) *X is locally connected.*
(5) *X is perfectly normal.*

7. Closed mappings and realcompactness. Frolík has proved the following:

THEOREM (F_2) [3][2]. *If φ is a perfect mapping from a realcompact, normal space X onto Y, then Y is realcompact.*

In this section, we shall deal with closed mappings and show, in Theorem 7.5, that the realcompactness is invariant under a closed mapping, in Theorem (F_2), if we replace "compactness of φ" by "local compactness of X". It seems to me that Theorem 7.5 is only one case for which the realcompactness is proved to be invariant under a closed mapping without any additional condition.

LEMMA 7.1. *If φ is a closed mapping from a normal space*

[2] It seems to me that the countable paracompactness is necessary.

X onto Y, then $\mathrm{cl}_{\beta X}\mathscr{L}_X\varphi^{-1}(y) = \mathscr{L}_{\beta X}\Phi^{-1}(y)$ *for every* $y \in Y$. *Furthermore, if* $\mathscr{L}_X\varphi^{-1}(y)$ *is compact, then* $\Phi^{-1}(y) - \varphi^{-1}(y)$ *is open-closed in* $\beta X - X$.

Proof. Since φ is closed, we have $\mathrm{cl}_{\beta X}\varphi^{-1}(y) = \Phi^{-1}(y)$ by 1.1 and 1.2. It is obvious that $\mathscr{L}_X\varphi^{-1}(y) \subset \mathscr{L}_{\beta X}\Phi^{-1}(y)$. Suppose that there is a point x in $\mathscr{L}_{\beta X}\Phi^{-1}(y) - \mathrm{cl}_{\beta X}\mathscr{L}_X\varphi^{-1}(y)$. We can find a nbd U(in βX) of x with $\mathrm{cl}_{\beta X}U \cap \mathrm{cl}_{\beta X}\mathscr{L}_X\varphi^{-1}(y) = \phi$. Since

$$\mathrm{cl}_{\beta X}\varphi^{-1}(y) = \Phi^{-1}(y),\ F = \mathrm{cl}_{\beta X}U \cap \varphi^{-1}(y) \neq \phi .$$

Next we shall show that $E = \mathrm{cl}_{\beta X}U \cap (X - \varphi^{-1}(y)) \neq \phi$. Since

$$x \in \mathscr{L}_{\beta X}\Phi^{-1}(y) - \mathrm{cl}_{\beta X}\mathscr{L}_X\varphi^{-1}(y) ,$$

U contains a point z of $\beta X - \Phi^{-1}(y)$, and hence, there is a nbd V (in βX) of z such that $V \subset U$ and $V \cap \Phi^{-1}(y) = \Phi$. X being dense in βX, V contains a point of $X - \varphi^{-1}(y)$. Thus $E \neq \phi$. Since

$$E \cap F = \mathrm{cl}_{\beta X}U \cap \varphi^{-1}(y) \cap (X - \varphi^{-1}(y)) = \phi$$

and X is normal, we have $\mathrm{cl}_{\beta X}E \cap \mathrm{cl}_{\beta X}F = \phi$. On the other hand, since $x \in \Phi^{-1}(y) = \mathrm{cl}_{\beta X}\varphi^{-1}(y)$ and U is a nbd of x, we have $\mathrm{cl}_{\beta X}F \ni x$ and $\mathrm{cl}_{\beta X}E \ni x$, i.e., $\mathrm{cl}_{\beta X}F \cap \mathrm{cl}_{\beta X}E \neq \phi$ which is a contradiction. The latter part is obvious.

In the following, $Y_c = \{y; y \in Y, \varphi^{-1}(y)$ is compact$\}$, $Y_0 = \{y; y \in Y$, $\mathscr{L}\varphi^{-1}(y)$ is compact but $\varphi^{-1}(y)$ is not compact$\}$ and $Y_1 = \{y; y \in Y$, $\mathscr{L}\varphi^{-1}(y)$ is not compact$\}$.

THEOREM 7.2.[3] *Let φ be a closed mapping from a locally compact, realcompact, normal space X onto Y; then we have*

(a) *$Y_0 \cup Y_1$ is closed.*

(b) *$Y - Y_1$ is locally compact.*

(c) *The closure of any neighborhood of y is not compact for every $y \in Y_1$.*

(d) *$Y_0 \cup Y_1$ is a discrete closed subset of Y.*

Proof. (*a*). Let $y \in Y_c$ be an accumulation point of $Y_0 \cup Y_1$. Since $\varphi^{-1}(y)$ is compact, there is a nbd V of $\varphi^{-1}(y)$ whose closure is compact. $M = Y - \varphi(X - V)$ is an open set containing y. Therefore there is a point $y' \in Y_0 \cup Y_1$ with $y' \in M$. This shows that

$$\varphi^{-1}(y') \subset \varphi^{-1}(M) \subset V \subset \bar{V}$$

and $\varphi^{-1}(y')$ is compact. This is a contradiction.

[3] This theorem is analogous to Theorem 4 in [**11**] in which X is locally compact, paracompact, normal space. The proofs of (a) and (b) are the very same as those given in [**11**].

(b). Let y be any point of $Y - Y_1$. Since $\mathscr{L}\varphi^{-1}(y)$ is compact, there is a nbd V of $\mathscr{L}\varphi^{-1}(y)$ whose closure is compact.

$$M = Y - \varphi(X - U)$$

is an open set containing y where $U = \varphi^{-1}(y) \cup V$. Then

$$\bar{M} \subset \overline{\varphi(U)} = \varphi(\bar{U}) = \varphi(\bar{V}) \cup \{y\}$$

is compact, and hence, $\bar{M}$ is compact. This shows that $Y - Y_1$ is locally compact.

(c). Suppose that there is a point $y \in Y_1$ which has a nbd W with the compact closure. Since $\mathscr{L}_X\varphi^{-1}(y)$ is not compact, there is a point $x \in \mathscr{L}_X\Phi^{-1}(y) - \mathscr{L}_{\beta X}\varphi^{-1}(y)$ by 7.1, and hence there is a function $f \in C(\beta X)$ with $0 \leqq f \leqq 1, f(x) = 0, f > 0$ on X by 1.8 since X is realcompact. We shall show that there is a sequence $\{z_n\}$ in

$$\varphi^{-1}(W) - \varphi^{-1}(y)$$

such that $\varphi(z_n) \neq \varphi(z_m)(n \neq m)$ and $\{f(z_n)\} \downarrow 0$. For

$$A_n = \{z\,; f(z) \leqq 1/n, z \in \varphi^{-1}(W) - \varphi^{-1}(y)\} \qquad (n = 1, 2, \cdots)\ ,$$

we have $x \in \mathrm{cl}_{\beta X}A_n$. If $\varphi(A_n)$ is finite, then $\varphi(A_n)$ does not contain y since φ is closed. On the other hand, since $y \in \mathrm{cl}_{\beta X}A_n$ and $y \in Y$, we have $y \in \Phi(\mathrm{cl}_{\beta X}A_n) \subset \mathrm{cl}_{\beta Y}\Phi(A_n) = \mathrm{cl}_{\beta Y}\varphi(A_n)$, and hence, $y \in Y \cap \mathrm{cl}_Y\varphi(A_n) = \varphi(A_n)$. Thus every A_n contains infinitely many points whose images, under φ, are distinct from each another. Therefore we have a desired sequence $\{z_n; X_n \in A_n\}$ (if necessary, take a suitable subsequence). Since $f > 0$ on X, $Z = \{z_n\}$ is a discrete closed subset. On the other hand, $\varphi(Z) \subset \bar{W}$ and $\bar{W}$ is compact, and hence, $\varphi(Z)$ has an accumulation point in $\varphi(Z)$. Let say $y_0 = \varphi(z_1)$ be such an accumulation point because φ is closed. X being normal, there is an open set U with $\varphi^{-1}(y_0) \subset U$ and $U \cap \{z_n; n = 2, 3, \cdots\} = \phi$.

$$M = Y - \varphi(X - U)$$

is an open set containing y_0 which is disjoint from a closed set

$$\varphi(Z) - \{y_0\} = \varphi(Z - \{z_1\})$$

because $Z - \{z_1\}$ is closed. This is a contradiction.

(d). We shall prove that every point of Y_1 is isolated in $Y_0 \cup Y_1$. If $\varphi^{-1}(y)$ has an open nbd U such that $\varphi(U) \cap (Y_0 \cup Y_1) = \{y\}$, then $M = Y - \varphi(X - U)$ is an open set with $(Y_0 \cap Y_1) \cap M = \{y\}$. This shows that every point of Y_1 is isolated in $Y_0 \cup Y_1$. Therefore, we can assume that there are a point $y \in Y_1$ and a point x in $\varphi^{-1}(y)$ such that any open nbd U of x has a compact closure and $\varphi(U) \cap (Y_0 \cup Y_1)$

contains infinitely many points $y_n(n=1, 2, \cdots)$ of $Y_0 \cup Y_1$. Let a_n be any point contained in $\varphi^{-1}(y_n) \cap U$. Then $\{a_n\}$ has an accumulation point a_0 in $\bar{U}$ because $\bar{U}$ is compact. Since $\varphi(a_n) = y_n \in Y_0 \cup Y_1$ and $Y_0 \cup Y_1$ is closed by (a), we have $y_0 = \varphi(a_0) \in Y_0 \cup Y_1$. Thus we can assume that there is a point $y_0 \in Y_0 \cup Y_1$ which is an accumulation point of $\{y_n; y_n \in Y_0 \cup Y_1\}$. Let $x'_n \in \Phi^{-1}(y_n) - \varphi^{-1}(y_n)$; then $\beta X - X$ being compact, $A \cap X = \phi$ where $A = \mathrm{cl}_{\beta X}\{x'_n\}$. If $A \cap \Phi^{-1}(y_0) = \phi$, then $y_0 \notin \Phi(A)$ which is impossible because $y_n \in \Phi(A)$ $(n = 1, 2, \cdots)$ and Φ is closed. Let $x'_n \in A \cap (\Phi^{-1}(y_0) - \varphi^{-1}(y_0))$ and f be a function of $C(\beta X)$ such that $0 \leqq f \leqq 1, f(x'_0) = 0$ and $f > 0$ on X by 1.8 because X is realcompact. Since $\mathrm{cl}_{\beta X}\varphi^{-1}(y) = \Phi^{-1}(y)$, without loss of generality, we can find a point x_n in $U_n \cap \varphi^{-1}(y_n)$ for every n such that $\{f(x_n)\} \downarrow 0$ where U_n is an open nbd (in βX) of x'_n. If $B \cap \varphi^{-1}(y_0) = \phi$ where $B = \mathrm{cl}_X \ \{x_n; n = 1, 2, \cdots\}$, then $\varphi(B) = \varphi(\bar{B}) = \overline{\varphi(B)} = \overline{\{y_n\}}$ does not contain y_0. This is impossible. Thus $B \cap \varphi^{-1}(y_0)$ contains a point x_0. It is obvious that $f(x_0) = 0$, but, this is a contradiction because $f > 0$ on X. Thus every point of Y_1 is isolated in $Y_0 \cup Y_1$.

Next we shall prove that every point y of Y_0 is isolated in $Y_0 \cup Y_1$, which shows that $Y_0 \cup Y_1$ is a discrete closed subset of Y.

$$\Phi_1 = \Phi \mid (\beta X - X)$$

is a closed mapping from a compact space $\beta X - X$ onto $\beta Y - Y_c$. For every $y \in Y_0$, $\Phi^{-1}(y) - \varphi^{-1}(y)$ is always open-closed by 7.1 in $\beta X - X$. Thus every point of Y_0 is isolated in $\beta Y - Y_c$, and hence, they are isolated in $Y_0 \cup Y_1 (\subset \beta Y - Y_c)$.
From (b) and (c) in 7.2, we have:

THEOREM 7.3. *Let φ be a closed mapping from a locally compact, realcompact, normal space X onto Y; then Y is locally compact if and only if $\mathscr{L}\varphi^{-1}(y)$ is compact for every $y \in Y$.*

This theorem is not necessarily true in general when X is locally compact normal, as shown by the following example by Prof. Morita. Let $X = [0, 1] \times W(\omega_1)$, $Y = [0, 1]$ and let φ be the projection: $X \to Y$. It is known that X and Y are both locally compact normal. Since Y is weakly separable and X is countably compact, φ is closed, but $\varphi^{-1}(\alpha)$ is not compact for every $\alpha \in Y$. Theorem 7.3 is also true, as shown in [11] replacing "realcompactness" by "paracompactness".

Under the assumption of 7.2, we shall consider the new space Z in the following way: we set up an equivalence relation "$\sim$" on X by the simple rule that "$x \sim x'$" if and only if both points x and x' belongs to the same $\varphi^{-1}(y)$ for some point $y \in Y_0 \cup Y_1$. Using this relation we define a space Z, that is, Z is a space obtained from X

by the topological identification (we notice that V of Z is open if and only if $\psi^{-1}(V)$ is open where ψ is the identification mapping). It is easy to see that $Z_c = \psi(X_c)$ is locally compact and homeomorphic with X_c, and $Z_0 \cup Z_1$ is a discrete closed subset where

$$X_c = \varphi^{-1}(Y_c),\ X_i = \varphi^{-1}(Y_i)(i = 0, 1),\ Z_0 = \psi(X_0)$$

and $Z_1 = \psi(X_1)$. ψ is obviously closed, and hence, Z is normal.

Now suppose that X is realcompact. $Z_0 \cup Z_1$ is realcompact as in the proof of realcompactness of V in 6.5 since $Z_0 \cup Z_1$ is closed and discrete. If every function of $C(Z)$ is continuously extended over a point z in $\beta Z - Z$, then there is a nbd U(in βZ) with $\mathrm{cl}_{\beta Z}\, U \cap (Z_0 \cup Z_1) = \phi$ because $Z_0 \cup Z_1$ is closed and realcompact. Thus $\mathrm{cl}_{\beta Z}\, U \cap Z_c \neq \phi$, but this is impossible since Z_c is homeomorphic with X_c. Therefore Z becomes a realcompact space.

Next we can construct a mapping λ from Z onto Y by the usual topological identification and it is easily seen that λ is perfect. Thus we have.

COROLLARY 7.4. *Let φ be a closed mapping from a realcompact, locally compact, normal space X onto Y; then φ admits a factorization $\varphi = \lambda\psi$ such that*

(i) *ψ is a closed mapping from X onto a realcompact normal space Z and $\{\psi^{-1}(z);\ z \in Z'\}$ is a closed discrete collection where Z' is the set of point z such that $\psi^{-1}(z)$ contains at least two points.*

(ii) *$\lambda: Z \longrightarrow Y$ is a perfect mapping.*

Since countable paracompactness is invariant under a closed mapping, we have the following theorem by 7.2 and Theorem (F_2).

THEOREM 7.5. *If φ is a closed mapping from a locally compact, countably paracompact, normal space X onto Y, then Y is realcompact when X is realcompact.*

8. **Examples.** Let M be a P-space and let K be a separable metric space. We denote by φ the projection: $M \times K \longrightarrow M$ and by Φ the Stone extension of φ from $\beta(M \times K)$ onto βM. Next ψ denotes the identity mapping on $M \times K$ and Ψ denotes the extension of ψ from $\beta(M \times K)$ onto $\beta M \times \beta K$ and let $\Psi_0 = \Psi \mid Z$ where

$$Z = \cup\{\Phi^{-1}(y);\ y \in M\} \subset \beta(M \times K).$$

LEMMA 8.1. (1) *The projection $\varphi: M \times K \longrightarrow M$ is closed.*

(2) *Z is realcompact if M is realcompact.*

(3) *Ψ_0 is a one-to-one mapping from Z onto $M \times \beta K$.*

(4) $\Psi^{-1}(M \times \beta K) = Z$.

Proof. (1). Let F be a closed subset of $M \times K$ and let $y \notin \varphi(F)$. Now suppose that y is not isolated. Since F is closed, for a point $(y, z) \in \varphi^{-1}(y)$, there is a nbd $W(y, z) = V(y) \times U(z)$ of (y, z) such that $W(y, z) \cap F = \phi$, where $V(y)$ and $U(z)$ are neighborhoods of y and z in M and K respectively. Since K is separable and $\{W(y, z); z \in K\}$ covers $\varphi^{-1}(y)$, there is a subcover $\{W(y, z_i); i = 1, 2, \cdots\}$. Let us put $V = \cap V_i$; then V is a nbd of y because y is a P-point, *and* hence, $V \times K$ is open and $(V \times K) \cap F = \phi$. This implies $y \notin \varphi(F)$ since $\varphi^{-1}(y) \subset V \times K$. Thus $\varphi(F)$ is a closed subset which shows the closedness of φ.

(2). Since Φ is closed and $\Phi^{-1}(y)$ is compact, Z is realcompact by 5.3.

(3) Since φ is closed, $\Phi^{-1}(y) = \mathrm{cl}_{\beta(M \times K)}\varphi^{-1}(y)$, and

$$\psi\varphi^{-1}(y) \subset \Psi_0(\Phi^{-1}(y)) .$$

On the other hand, $\psi\varphi^{-1}(y) = \{y\} \times K$ is dense in $\{y\} \times \beta K$. This implies that $\Psi_0(\Phi^{-1}(y)) = \{y\} \times \beta K$, equivalently $\Psi_0^{-1}(\{y\} \times \beta K) = \Phi^{-1}(y)$ because $\Phi^{-1}(y)$ is compact. Thus $\Psi_0(Z) = M \times \beta K$, that is, Ψ_0 is onto.

Next we shall show that Ψ_0 is one-to-one. Suppose that there are a point $y^* \in (\{y\} \times \beta K) - (\{y\} \times K)$ and $x_1, x_2 \in \Psi_0^{-1}(y^*)$, $x_1 \neq x_2$. There are open sets V_1(in Z) and V_2(in Z) of x_1 and x_2 respectively with $\bar{V}_1 \cap \bar{V}_2 = \phi$. Let us put $F_i = \bar{V}_i \cap \varphi^{-1}(y)$; then $F_i \neq \phi$ since $\Phi^{-1}(y) = \mathrm{cl}_{\beta(M \times K)}\varphi^{-1}(y)$. Since $\mathrm{cl}_{M \times \beta K}\psi(F_i) \subset \{y\} \times \beta K$, F_i is a closed subset of a normal space $\{y\} \times K$ and $\beta(\{y\} \times K) = \{y\} \times \beta K$, we have $\mathrm{cl}_{M \times \beta K}\psi(F_1) \cap \mathrm{cl}_{M \times \beta K}\psi(F_2) = \phi$. On the other hand,

$$x_i \in \mathrm{cl}_{\beta(M \times K) \cap Z}F_i \subset \Phi^{-1}(y)$$

implies that $y^* \in \Psi_0(\mathrm{cl}_{\beta(M \times K) \cap Z}F_i \subset \mathrm{cl}_{M \times \beta K}\Psi_0(F_i) = \mathrm{cl}_{M \times \beta K}\psi(F_i)$ (i = 1, 2). This is a contradiction.

(4). Suppose that there is a point $w \in \beta(M \times K) - Z$ such that $\Psi(w) = (y, \alpha) \in M \times \beta K$. There are open subsets V_1 and V_2 in $\beta(M \times K)$ such that $w \in V_2$, $\Phi^{-1}(y) \subset V_1$ and $\bar{V}_1 \cap \bar{V}_2 = \phi$. $\bar{V}_2 \cap Z$ is not empty and $\Phi(\bar{V}_2 \cap Z)$ is a subset of M containing y. Since

$$\Psi_0^{-1}(\{y\} \times \beta K) = \Phi^{-1}(y)$$

by (3), we have $\Psi_0(\bar{V}_2 \cap Z) \cap (\{y\} \times \beta K) = \phi$. Let μ be the projection: $M \times \beta K \rightarrow M$; then, we have $\Phi(A) = \mu\Phi_0(A)$ for every subset A of Z because $\Psi_0^{-1}(\{y\} \times \beta K) = \Phi^{-1}(y)$. Thus

$$\Phi(\bar{V}_2 \cap Z) = \mu\Psi_0(\bar{V}_2 \cap Z) \not\ni y$$

which is a contradiction, and hence, we have $Z = \Psi^{-1}(M \times \beta K)$.

Let M be a realcompact nondiscrete P-space; then $M \times K$ is realcompact and there is a function f of $C(\beta(M \times K))$ such that $f > 0$ on $M \times K$ and $Z(f) \cap \Phi^{-1}(y) \neq \phi$ for a given nonisolated point y of M. We notice $\Phi^{-1}(y) \operatorname{cl}_{\beta(M \times K)} \varphi^{-1}(y) = \operatorname{cl}_Z \varphi^{-1}(y)$ $(\cong \{y\} \times \beta K)$. In the following, we put $A_y = \Phi^{-1}(y)$ and $B_y = A_y - \varphi^{-1}(y)$.

Next we shall show that we cannot replace a Z-mapping by an open WZ-mapping in Theorem 5.3.

EXAMPLE 8.2. $X = Z - (Z(f) \cap B_y)$ is not realcompact and a mapping $\lambda = \Phi \mid X$ is an open WZ-mapping from X onto M and λ is not a Z-mapping.

Proof. It is obvious that Φ is open and closed, X is open in Z, $\Phi^{-1}(y') = \lambda^{-1}(y')$ for every y' $(\neq y)$ and

$$\varphi^{-1}(y) \subset \lambda^{-1}(y) = A_y - Z(f) \cap B_y)) ,$$

and hence λ is an open WZ-mapping. Thus to prove 8.2, it is sufficient to show that X is not realcompact by 5.3. Suppose that X is realcompact, then there are a function $h \in C(X)$ and a point $x^* \in B_y$ such that h can not be continuously extended over x^*. Since every subset $\lambda^{-1}(y') = \Phi^{-1}(y')$ is compact for $y' \neq y$, h is bounded on $\lambda^{-1}(y')$. If h is bounded on a $W \cap X$ where W is a nbd (in $\beta(M \times K)$) of x^*, then h is continuously extended over x^*. Thus for every nbd W of x^*, h is not bounded on $W \cap X$. Without loss of generality, we can assume that h is nonnegative on X. Therefore, for every n, there is a nbd W_n(in $\beta(M \times K)$) of x^* with $h \geqq n$ on $W_n \cap X$. $\varphi^{-1}(y) \cap W_n$ contains a point (y, k_n), and hence there are neighborhoods O_n and Q_n of y and k_n respectively such that $h \geqq n$ on $O_n \times Q_n$. Since y is a P-point, $V = \cap O_n$ is a nbd of y and h is not bounded on

$$A = \{(y_0, k_n); n = 1, 2, \cdots\}$$

where y_0 is some point of V and $y \neq y_0$. On the other hand, h is bounded on A and $A \subset \Phi^{-1}(y_0)$. This is a contradiction. Thus X is not realcompact.

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