A NOTE ON DAVID HARRISON’S THEORY OF PREPRIMES

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A Stone ring is a partially ordered ring \( K \) with unit element \( 1 \) satisfying (1) \( 1 \) is positive; (2) for every \( x \) in \( K \) there exists a natural number \( n \) such that \( n \cdot 1 - x \) belongs to \( K \); and (3) if \( 1 + nx \) is positive for all natural numbers \( n \) then \( x \) is positive. Our first theorem: Every Stone ring is order-isomorphic with a subring of the ring of all continuous real functions on some compact Hausdorff space, with the usual partial order. A corollary is a theorem first proved by Harrison: Let \( K \) be a partially ordered ring satisfying conditions (1) and (2), and suppose the positive cone of \( K \) is maximal in the family of all subsets of \( K \) which exclude \(-1\) and are closed under addition and multiplication. Then \( K \) is order-isomorphic with a subring of the reals.

The present paper is inspired by David Harrison's recently begun program of arithmetical ring theory where the basic objects are primes and preprimes; the positive cones of a ring are example of preprimes.

Throughout the paper, \( K \) will be a ring with unit element \( 1 \), and \( N \) will denote the set of positive integers. A preprime \( P \) in \( K \) is a nonempty subset of \( K \) excluding \(-1\) and closed under addition and multiplication. A prime in \( K \) is a preprime maximal relative to set inclusion. A preprime \( P \) is infinite provided it contains both zero and \( 1 \), and is conic if \( P \cap ( -P ) = \{ 0 \} \). A conic preprime is simply a positive cone and induces a partial order: \( x \geq y \iff y \leq x - y \in P \).

A preprime \( P \) is Archimedean if for all \( x \) in \( K \) there exists a natural number \( n \) with \( n - x \) in \( P \), (condition (2) in the definition of Stone ring) and is (AC) if from \( 1 + nx \in P \) for all \( n \in N \) follows \( x \in P \) (condition (3)). We redefine a Stone ring as a pair \( \langle K, P \rangle \) where \( P \) is an infinite conic Archimedean (AC) preprime in \( K \). An imbedding of \( \langle K, P \rangle \) in \( \langle K', P' \rangle \) is an injective ring homomorphism \( \psi: K \to K' \) such that \( P = \psi^{-1}(P') \). If \( X \) is a compact Hausdorff space, \( C(X) \) denotes the ring of all continuous real functions on \( X \), \( P(X) \) denotes the subset of nonnegative functions. If \( K \) is any subring of \( C(X) \) then \( \langle K, K \cap P(X) \rangle \) is a Stone ring. The principal tool in the proof of Theorem 1 is the Stone-Kadison ordered algebra theorem [3; Theorem 3.1], which characterizes \( C(X) \) as a complete Archimedean ordered algebra. To imbed a Stone ring \( \langle K, P \rangle \) in such an algebra we show that \( K \) is torsionfree, imbed it in a divisible ring \( K_N \), put a norm on \( K_N \) and then complete it to \( K^* \). At each step we have an imbedding of Stone rings:
where the last is Kadison’s order-isomorphism. If $P$ is a prime then so is $P_\pi$.  [An order-isomorphism is an imbedding onto.]

In the proofs following, $\langle K, P \rangle$ is a Stone ring, $N$ is the set of all positive integers.

**Proposition 1.** If $n \in N$, $a \in K$, and $na \geq 0$ then $a \geq 0$.

**Proof.** By the unique factorization in $N$, it is enough to prove the proposition for the case where $n$ is a prime number. Suppose for all primes $q < p$ and all $a \in K$, $qa \geq 0$ implies $a \geq 0$. Then for all $n < p$ and all $a \in K$, $na \geq 0$ implies $a \geq 0$. Now suppose that $pa \geq 0$ but $a \not\geq 0$. By the Archimedean property choose $m$ in $N$ with $m + a \geq 0$, $x = m - 1 + a \not\geq 0$. Then $px \geq 0$, $1 + x \geq 0$ and for all $n$ in $N$, $1 + (pn + d)x \geq 0$, if $d = 0$ or $d = 1$. [In case $p = 2$ this implies that $1 + kx \geq 0$ for all $k$, so $x \geq 0$ by (AC), a contradiction; hence $2a \geq 0$ implies $a \geq 0$.] Now let $1 < d < p$, with $d$ in $N$. Since $p$ is a prime there exists $e$ in $N$, with $1 < e < p$, $ed = 1 + pn$, for some $n$ in $N$. Then $e(1 + dx) = e + (1 + pn)x = (e - 1) + (1 + x) + (pnx) \geq 0$. Since $e < p$ this implies that $1 + dx \geq 0$. So for all $k$ in $N$, $1 + (pk + d)x = 1 + dx + pkx \geq 0, 0 \leq d \leq p - 1$. That is, $1 + nx \geq 0$ for all $n$ in $N$. By (AC) again, $x \geq 0$, a contradiction. So $a \geq 0$ and the induction is complete.

Now put

$$K_\pi = Q \otimes K = \{k/n; k \in K, n \in N\},$$

$$P_\pi = \{p/n; p \in P, n \in N\},$$

$$\varphi: K \rightarrow K_\pi, \varphi(k) = k/1.$$

**Proposition 2.** $\langle K_\pi, P_\pi \rangle$ is also a Stone ring. If $P$ is a prime then so is $P_\pi$. $\varphi$ is an imbedding.

**Proof.** That $\varphi$ is injective follows from Proposition 1. If $k/n$, for $k$ in $K, n$ in $N$, belongs to $P_\pi$, then $k$ belongs to $P$. For $k/n$ in $P_\pi$ implies $k/n = p/m$, for some $p$ in $P, m$ in $N$, so $mk = np \in P$. By Proposition 1, $k \in P$. Hence $\varphi$ is an imbedding. The preprime, infinite, and conical properties of $P_\pi$ follow easily from the corresponding properties for $P$. For the Archimedean property, let $k/m$ be arbitrary in $K_\pi (k \in K, m \in N)$ and choose $n$ in $N$ with $n > k$. Then $n - k/m = (nm - k)/m$ belongs to $P_\pi$ since $nm > k, m \in N$. Now if $1 + n(k/m) \geq 0$ holds in $K_\pi$, with $m$ in $N, k$ in $K$, and for all $n$ in $N$, then for all $n, \varphi(1 + nk) = 1 + mn(k/m) \in P_\pi$. Since $\varphi$ is an imbedding, $1 + nk \in P$. By the (AC) property for $P, k \in P, k/m \in P_\pi$. 

$$\langle K, P \rangle \rightarrow \langle K_\pi, P_\pi \rangle \rightarrow \langle K^*, P^* \rangle \rightarrow \langle C(X), P(X) \rangle,$$
This establishes (AC) for $P_N$. Finally let $P'$ be a preprime containing $P_N$ and let $P = \varphi^{-1}(P')$. Then $P$ is a preprime containing $P$. If the first containment is proper so is the second. This proves that if $P$ is a prime then $P_N$ is a prime.

**NOTE.** The additive group of $K_N$ is divisible. If $K$ were already divisible then $\varphi$ would be an order-isomorphism of $\langle K, P \rangle$ onto $\langle K_N, P_N \rangle$. The rational multiples of 1 in $K_N$ form a field order-isomorphic with $Q$.

Now define $t$ on $K_N$ by

$$t(x) = \inf \{r; -r < x < r, r \in \mathbb{Q}\}.$$

**Proposition 3.** The function $t$ is a norm on $K_N$:

(a) $t(x) \geq 0$; $t(x) = 0$ if and only if $x = 0$.

(b) $t(x + y) \leq t(x) + t(y)$.

(c) $t(xy) \leq t(x)t(y)$

(d) $t(rx) = |r|t(x)$ for $r$ in $Q$.

Put $K^*$ equal to the completion of $K_N$, $P^*$ equal to the closure of $P_N$ in $K^*$. Then $\langle K^*, P^* \rangle$ is a Stone ring and an Archimedean ordered algebra as defined by Kadison.

**Proof.** The property (a) follows from (AC). Properties (b) and (c) follow from: if $-r < x < r, -s < y < s$ then $-(r + s) < x + y < r + s$, and $-rs < xy < rs$. See [1], §2. The proofs there make no use of commutativity or of multiplicative inverses. Property (d) is a consequence of: $-r < x < r$ if and only if $-rq < qx < rq$, where $q$ is a positive rational. It is clear that $t(-x) = t(x)$ and for rational $r$, $t(r) = |r|$. We now identify $K_N$ with its injection in its completion $K^*$ and note that $P^* \cap K_N = P_N$: for if $k \in P^* \cap K_N$ then $k = \lim p_n$, $p_n \in P_N$, and $p_n$ may be chosen so that $-1/n < k - p_n < 1/n$ for all $n \in N$; it follows that $1 + nk > np_n > 0$ for all $n \in N$ and thence by (AC) that $k \in P_N$. The reverse inclusion is obvious. It remains to prove that $P^*$ is an infinite conical Archimedean (AC) preprime. It is certainly closed under addition and multiplication. Let $x \in P^* \cap (-P^*)$. Then there exist positive sequences $p_n$ and $q_n$ with $x = \lim p_n$, $-x = \lim q_n$, $0 = \lim (p_n + q_n)$. Thus if $\varepsilon$ is any positive real then for all large $n$, $0 \leq p_n \leq p_n + q_n < \varepsilon$, so $x = \lim p_n = 0$; $P^*$ is therefore conical. Let $x_n \in K_N$, with $x = \lim x_n$. The Cauchy sequence $\{x_n\}$ is bounded in norm so there exists an integer $m$ with $m > x_n$ for all $n$. Hence $m - x = \lim (m - x_n) \in P^*$, $m > x$. This shows $P^*$ is Archimedean. Now let $1 + 2x \in P^*$ for all $n$ in $N$ ($x \in K^*$). $P^*$, as closure of $P$, is closed and hence contains $x = \lim (x + 1/n)$, since $x + 1/n$ belongs to $P^*$. Thus $P^*$ is (AC). That $1 \in P^*$ and $-1 \in P^*$ are obvious, and
it has now been proved that \( \langle K^*, P^* \rangle \) is a Stone ring. The closure of \( Q \) in \( K^* \) is (order-isomorphic with) the reals \( R \). Using \( t \) for the induced norm in \( K^* \) we have

\[
(e) \quad t(rx) = |r| t(x) \quad \text{for all} \quad r \in R.
\]

\( R \) is contained in the center of \( K^* \) and so \( \langle K^*, P^* \rangle \) is an algebra over the reals. For the sake of completeness we list Kadison’s axioms for an Archimedean ordered algebra. Each is obviously satisfied by \( \langle K^*, P^* \rangle \) with \( e = 1 \).

1. \( K^* \) is a real algebra with unit \( e \).
2. \( P^* \) is closed under addition, multiplication, and multiplication by positive reals.
3. For every \( x \) in \( K^* \) there exists a positive real \( r \) with \( re > x \).
4. If \( re \geq x \) for all positive real \( r \), then \( x \leq 0 \).

An Archimedean ordered algebra is *complete* if and only if it is complete in our norm \( t \). Thus \( \langle K^*, P^* \rangle \) is a complete Archimedean ordered algebra. Collecting results of Propositions 1, 2, and 3 and applying Theorem 3.1 of Kadison we get our Theorem 1.

Now we are ready to prove the corollary. As we remarked earlier, Harrison showed that a prime \( P \) satisfying the hypotheses there is also (AC). By Proposition 2, \( P_\mathcal{N} \) is also a prime. Now identify each of \( \langle K, P \rangle, \langle K_\mathcal{X}, P_\mathcal{X} \rangle, \langle K^*, P^* \rangle \) with its imbedding in \( \langle C(X), P(X) \rangle \), so that \( P(P_\mathcal{X}) \) is the set of all nonnegative functions in \( K(K_\mathcal{X}) \). The proof is completed by showing that \( X \) is a singleton. Suppose that \( x \) and \( y \) are distinct points of \( X \). Since \( X \) is normal and \( K_\mathcal{X} \) is dense in \( C(X) \), Urysohn’s lemma guarantees that there is a function \( f \) in \( K_\mathcal{X} \) with \( f(x) > 0, f(y) < 0 \). Then \( P' = \{ g; g \in K_\mathcal{X} \text{ and } g(x) \geq 0 \} \) is a preprime in \( K_\mathcal{X} \) containing \( P_\mathcal{X} \) and \( f \), while \( f \) is not in \( P_\mathcal{X} \). This contradicts the primality of \( P_\mathcal{X} \) and the corollary is proved.

**Two Examples.** 1. Example of a ring \( \langle K, P \rangle \) where all the conditions of Theorem 1 hold for \( P \) except the Archimedean condition. Let \( K \) be the ring of all \( 2 \times 2 \) real matrices, \( P \) the set of matrices with every entry nonnegative.

2. Example of a ring \( \langle K', P' \rangle \) where \( P' \) satisfies all except the condition (AC). Put \( K' \) equal to the set of all triangular \( 2 \times 2 \) matrices over \( R \) and let \( P' \) be the subset consisting of 0 and all matrices with strictly positive diagonal entries. Thus if either of the Archimedean conditions is omitted then commutativity cannot be deduced.
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