SURFACES HARMONICALLY IMMERSED IN $E^3$

TILLA WEINSTEIN
SURFACES HARMONICALLY IMMERSED IN $E^3$

Tilla Klotz

In this paper we study surfaces in $E^3$ which satisfy conditions necessary and/or sufficient to insure their harmonic immersion with respect to a fixed but not necessarily ordinary conformal structure. Our consideration of such surfaces is based upon the notion that surfaces which share some essential property of minimal surfaces are bound to be interesting. Thus our use here of nonstandard conformal structures is simply a device for the identification of such a class of surfaces distinct from others already much studied, such as quasi-minimal surfaces or surfaces of constant mean curvature. In the end, any such endeavor serves to distinguish those facts about minimal surfaces which are special to them from among the many facts which apply to larger classes of surfaces sharing some one vital property of minimal surfaces.

The more quotable results in this paper refer to a conformal structure $R$, determined by a fixed positive definite linear combination $A = fI + gII$ of the fundamental forms on the surface, with $f$ and $g$ smooth functions. Specifically, we show that mean curvature $H$ cannot be bounded away from zero on a complete $R$-harmonically immersed surface in $E^3$. This result is less general than it might seem. For we also prove that where $H \neq 0$ on an $R$-harmonically immersed surface, $A \propto II'$, with $II'$ defined by $\sqrt{H^2 - KII'} = HII - KI$. Included is an example of an $R$-harmonically immersed surface on which $H \neq 0$.

1. This section contains an explanation of our notation. Throughout the paper $X: S \to E^3$ denotes the $C^k$ immersion of an oriented $C^k$ surface $S$ in $E^3$, with $k \geq 2$. At certain points below we make the stronger assumption that $k \geq 4$.

A conformal structure $R$ is said to be defined on $S$ if $R$ is a Riemann surface on the point set $S$ each of whose conformal parameters $w = u + iv$ yields a pair of $C^j$ coordinates $u, v$ on $S$, with $2 \leq j \leq k$. At certain points below we need the stronger assumption that $3 \leq j \leq k$. Given a conformal structure $R$ on $S$, we call $u, v$ $R$-isothermal coordinates on $S$ provided that $u + iv = w$ is a conformal parameter on $R$. And quantities on $S$ evaluated only for all possible $R$-isothermal coordinates are referred to as quantities on $R$. Thus, we say that $S$ is $R$-harmonically immersed in $E^3$ by $X$ if and only if $\Delta X \equiv 0$ on $R$, that is, if and only if

$$X_{uu} + X_{vv} \equiv 0$$
for \( R \)-isothermal coordinates \( u, v \) anywhere on \( S \).

Specifically alluded to below are the conformal structures \( R_i \) determined by the first fundamental form \( I \) on \( S \); \( R_j \) determined (if Gauss curvature \( K > 0 \) and mean curvature \( H > 0 \) on \( S \)) by the second fundamental form \( II \) on \( S \); and \( R'_r \) determined (if \( K < 0 \) on \( S \)) by the form

\[
II' = \frac{1}{\sqrt{H^2 - K}}(HII - KI)
\]
on \( S \). More generally, \( R_i \) denotes the conformal structure on \( S \) determined by any fixed positive definite form \( A \) on \( S \) which is a linear combination of \( I \) and \( II \), so that

\[
A = fI + gII
\]
with \( f \) and \( g \) \( C^{k-2} \) real valued functions on \( S \). Since \( R_i \)-isothermal coordinates on \( S \) need only be \( C^j \) with \( j = k - 1 \), \( k \geq 4 \) will be assumed wherever \( j \geq 3 \) is required (as, for example, in using the Codazzi Mainardi equations on \( R_i \)). There are, of course, on any \( S \) conformal structures \( R \) which are not of the form \( R_i \).

Given an arbitrary conformal structure \( R \) on \( S \), the expressions

\[
\Omega_1 = ((E - G) - 2iF)dw^2
\]
\[
\Omega_2 = ((L - N) - 2iM)dw^2
\]

associated with the fundamental forms

\[
I = Edu^2 + 2Fdudv + Gdv^2
\]
\[
II = Ldu^2 + 2Mdudv + Ndv^2
\]
on \( R \), are quadratic differentials on \( R \) ([2], p. 1285). We will use below the formal differential operators

\[
\frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)
\]
on \( R \). An arbitrary quadratic differential

\[
\Omega = \phi dw^2
\]
on \( R \) is said to be \( R \)-holomorphic if and only if

\[
\phi_{\bar{w}} \equiv 0
\]
for every conformal parameter \( w \) on \( R \).

2. Results of a local nature are presented in this section.
**Lemma 1.** \( X: S \rightarrow E^3 \) is \( R \)-harmonic if and only if \( \Omega_i \) is \( R \)-holomorphic while \((L + N) = 0\) on \( R \).

**Proof.** The definition of \( I \) yields

\[
\Omega_i = 4(X_w \cdot X_{\bar{w}})dw^2
\]

so that \( \Omega_i \) is \( R \)-holomorphic if and only if

\[
X_w \cdot X_{\bar{w}} = 0,
\]

that is, equivalently, if and only if

\[
\Delta X \cdot X_u = \Delta X \cdot X_v = 0
\]

for every conformal parameter \( w = u + iv \) on \( R \). But the Gauss equations ([6], p. 107) give \( \Delta X \) as a linear combination of the linearly independent vectors \( X_u, X_v \) and the unit normal \( \hat{X} \), with \((L + N)\) the coefficient of \( \hat{X} \). Thus

\[
(1) \quad \Delta X = (L + N)\hat{X}
\]

on \( R \) if and only if \( \Omega_i \) is \( R \)-holomorphic. Moreover, \( \Delta X \equiv 0 \) on \( R \) if and only if \( \Omega_i \) is \( R \)-holomorphic, while \((L + N) \equiv 0\) on \( R \).

**Lemma 2.** If \( X: S \rightarrow E^3 \) is \( R \)-harmonic, then

(i) \( K \leq 0 \) on \( S \)

(ii) \( K = 0 \) only where \( H = 0 \), i.e. only at flat points, and

(iii) \( R \equiv R_i \), or else the points at which \( R = R_i \) are isolated.

**Proof.** First, if \( X: S \rightarrow E^3 \) is \( R \)-harmonic then

\[
(2) \quad K = -\frac{(L^2 + M^2)}{EG - F^2} \leq 0,
\]

on \( R \), since \((L + N) \equiv 0\) on \( R \), thus establishing (i). Next, (2) shows that \( K = 0 \) on \( S \) only where \( L = M = -N = 0 \) on \( R \), thus establishing (ii). Finally, since the zeros of \( \Omega_i \) on \( R \) are those points at which \( E = G \) and \( F = 0 \) on \( R \), we conclude that \( R \) and \( R_i \) coincide (in assigning identical angle measurements) precisely at the zeros of \( \Omega_i \). But the zeros of an \( R \)-holomorphic quadratic differential \( \Omega \) must be isolated unless \( \Omega \equiv 0 \) on \( R \), thus establishing (iii).

**Lemma 3.** If \( \Omega_i \) is \( R \)-holomorphic, there exists wherever \( R \neq R_i \) a function

\[
cosh^{-1}(2E - 1) > 0
\]
which is \( R \)-subharmonic where \( K \leq 0 \), \( R \)-superharmonic where \( K \geq 0 \), and constant only if \( K = 0 \).

**Proof.** Since \( \Omega_1 \) is \( R \)-holomorphic, there exist near any point at which \( \Omega_1 \neq 0 \) special \( R \)-isothermal coordinates \( u, v \) in terms of which \( \Omega_1 = dw^s \) ([1], p. 103), so that

\[
(3) \quad I = Edu^s + (E - 1)dv^s.
\]

Since such coordinates \( u, v \) are uniquely determined up to additive constants and/or simultaneous multiplication by 1 or \(-1\), the function \( E > 1 \) is well defined on \( S \) wherever \( \Omega_1 \neq 0 \) on \( R \), i.e. wherever \( R \neq R_i \). The Theorema Egregium equation applied to (3) yields

\[
(4) \quad K = \frac{1}{2\sqrt{E(E - 1)}} \Delta \cosh^{-1}(2E - 1).
\]

Thus \( \Delta \cosh^{-1}(2E - 1) \geq 0 \) on \( R \) where \( K \leq 0 \), and \( \Delta \cosh^{-1}(2E - 1) \leq 0 \) on \( R \) where \( K \geq 0 \). If \( \cosh^{-1}(2E - 1) \) is constant, (4) makes \( K = 0 \).

**Remark.** By arguments used to prove the following lemma, the function \( E > 1 \) just defined is itself \( R \)-subharmonic where \( R \neq R_i \) and \( K \leq 0 \) on \( S \), provided that \( \Omega_1 \) is \( R \)-holomorphic.

**Lemma 4.** If \( X: S \rightarrow E^3 \) is \( R \)-harmonic, there exists wherever \( R \neq R_i \) an \( R \)-subharmonic function \( E > 1 \) which is constant only if \( S \) is immersed as a portion of a plane.

**Proof.** No claims are made unless \( R \neq R_i \) somewhere on \( S \). Since \( X: S \rightarrow E^3 \) is assumed to be \( R \)-harmonic, Lemma 1 states that \( \Omega_1 \) is \( R \)-holomorphic, so that the function \( E \) referred to in Lemma 3 is available on \( S \) wherever \( R \neq R_i \). Equation (4) may be rewritten to read

\[
(5) \quad \Delta E = \frac{2E - 1}{2E(E - 1)} \{E_u^2 + E_v^2 \} - 2E(E - 1)K,
\]

giving \( \Delta E \geq 0 \) for \( E > 1 \) on \( R \) where \( R \neq R_i \), since \( K \leq 0 \) by Lemma 2. If \( E \) is constant, it follows from (4) using Lemma 2 that \( K = H = 0 \) wherever \( R \neq R_i \), a relation which extends easily by continuity to the isolated points on \( S \) at which \( R = R_i \).

**Lemma 5.** If \( \Omega_1 \) is \( R_i \)-holomorphic with \( A = fI + gII \), then where \( R \neq R_i \) there exist, locally, special \( R_i \)-isothermal lines-of-curvature coordinates \( u, v \) in terms of which

\[
(6) \quad g(k_2 - k_i)I = (f + gk_2)dw^s + (f + gk_i)dv^s,
\]
while the principal curvatures $k_1$ and $k_2$ cannot be equal.

Proof. Where $R_4 \neq R_1$, special $R_4$-isothermal coordinates $u, v$ may be locally introduced in terms of which $I$ has the form (3) while

$$\Delta = \mu (d\nu^2 + dv^2)$$

with $\mu > 0$. Moreover, since $g \neq 0$, it is easy to check that $M = 0$, making $u, v$ lines-of-curvature coordinates as well, so that

$$\Pi = k_1 E du^2 + k_2 (E - 1) dv^2.$$ 

Now (7) yields

$$0 < \mu = E (f + gk_1) = (E - 1) (f + gk_2),$$

making $k_1 = k_2$ impossible since $E > 1$ and $g \neq 0$. Finally, (6) follows by simple arithmetic. Note that here,

$$\frac{f + gk_1}{g(k_2 - k_1)}$$

is the expression $E$ referred to in Lemmas 3 and 4.

Lemma 6. If $X: S \to E^3$ is $R_4$-harmonic, then

$$R_4 = R'_2$$

where $R_4 \neq R_1$, so that $R_4 = R_1$ precisely where $H = 0$, making zeros of $H$ isolated unless $H \equiv 0$.

Proof. By Lemma 1, $\Omega_1$ is $R_r$-holomorphic, and where $R_4 \neq R_1$, we may introduce the special $R_r$-isothermal lines-of-curvature coordinates provided by Lemma 5, in terms of which (trivially)

$$HM = KF = 0$$

while, by Lemma 1,

$$L = -N.$$ 

That $L \neq 0$ follows since, by Lemma 5, $K = H = 0$ is possible only where $R_4 = R_1$. Thus Lemma 4 of [2] indicates that our coordinates are $R'_2$-isothermal, making $R_4 = R'_2$ wherever $R_4 \neq R_1$. Of course, $H \neq 0$ where $R_4 \neq R_1$, since $L \neq 0$ and

$$H = \frac{L}{2E(1 - E)}.$$ 

On the other hand, it is well known that $H = 0$ if $\Delta X = 0$ on $R_1$ at
EXAMPLE 1. There is (at least locally) an $R'$-harmonic imbedding $X: S \to E^3$ with $H \neq 0$. As an example, take for $S$ any portion of the $u, v$ plane, and require that $X$ produce the fundamental forms

$$I = \left(\frac{e^u + 2}{2}\right)du^2 + \left(\frac{e^u}{2}\right)dv^2$$

$$II = \frac{1}{2} \sqrt{\frac{e^u}{2 + e^u}} (du^2 - dv^2).$$

Since the Codazzi-Mainardi and Theorema Egregium equations are satisfied, the fundamental existence theorem of surface theory gives the local existence of $X$. On the other hand, the requirements of Lemma 1 are fulfilled with $H \neq 0$.

EXAMPLE 2. It is an easy matter to find $R$-harmonic imbeddings $X: S \to E^3$ for an $R$ not of the form $R_1$. If $S$ is the $x, y$-plane, consider the imbedding $X: S \to E^3$ defined by

$$X(x, y) = (x, y, xy).$$

Use for $R$ the conformal structure determined by the metric $dx^2 + dy^2$ on $S$. The asymptotic coordinates $x, y$ on the imbedded surface are not $R_1$-isothermal, nor are they $R'_2$-isothermal by the Remark, p. 1284 of [2]. Lemma 6 thus establishes that $R$ is not of the form $R_1$. We note in passing that $\Omega_z$ happens also to be $R$-holomorphic for this $X: S \to E^3$.

REMARK. Using Lemma 6 and the facts on p. 1284 of [2], it is easy to check that no immersion $X: S \to E^3$ of the form

$$X(x, y) = (x, y, f(x, y))$$

with $S$ a portion of the $x, y$-plane can be $R_1$-harmonic with $x, y$ $R_1$-isothermal coordinates unless $f \equiv c$, i.e. unless $S$ is immersed as a piece of plane.

LEMMA 7. If $X: S \to E^3$ is $R'$-harmonic, then near any point at which $R'_2 \neq R_1$ there exist special $R'_2$-isothermal coordinates $u, v$ in terms of which

$$(k_1 + k_2)I = k_3 dw^2 - k_4 dv^2$$

$$(k_1 + k_2)II = k_3 k_4 (dw^2 - dv^2).$$

Proof. Use Lemma 5, setting
\[ f = \frac{-K}{\sqrt{H^2 - K}} , \quad g = \frac{H}{\sqrt{H^2 - K}} . \]

Note that by (8), \( k_2 \) has the sign of \( H \) on \( S \), so that \( |k_2| > |k_1| \). Here, of course,

\[
\frac{k_2}{k_1 + k_2}
\]

is the function \( E \) referred to in Lemma 4.

3. Results in the large are presented in this section.

**Theorem 1.** If \( K \geq 0 \) on a complete \( S \) in \( E^3 \) while \( \Omega_1 \) is \( R \)-holomorphic and never vanishes on \( R \), then \( K \equiv 0 \) on \( S \).

**Proof.** If \( K \neq 0 \) somewhere on \( S \), then by Lemma 5 of [4], \( S \) is simply connected. Next, since an \( R \)-holomorphic quadratic differential must vanish identically if \( R \) is (conformally) a sphere, we can apply Lemma 5 of [4] once more to conclude that \( R \) is (conformally) the finite plane. Finally, since \( K \geq 0 \) on \( S \), Lemma 3 above yields the \( R \)-superharmonic function \( \cosh^{-1}(2E - 1) > 0 \) which must be constant on \( R \). Thus \( E \) is constant, and using (4), \( K \equiv 0 \) on \( S \).

Theorem 1 includes the elementary fact that there is no complete, umbilic free \( S \) in \( E^3 \) on which \( K \equiv c > 0 \). For, as shown in [3], \( K \equiv c > 0 \) on \( S \) in \( E^3 \) if and only if \( \Omega_1 \) is \( R \)-holomorphic with zeros of \( \Omega_1 \) on \( R \) corresponding to umbilics on \( S \). In the following theorem, \( \Lambda = fI + gII \) as usual, while the principal curvatures on the surface \( S \) in question are numbered so that \( |f + gk_2| > |f + gk_1| \) everywhere.

**Theorem 2.** If \( K \leq 0 \) on a complete \( S \) in \( E^3 \) while \( \Omega_1 \) is \( R_3 \)-holomorphic and never vanishes on \( R_3 \), then

\[
(9) \quad \frac{f + gk_2}{g(k_2 - k_1)}
\]

cannot be bounded from above unless \( K \equiv 0 \) on \( S \).

**Proof.** Since \( \Omega_1 \) is \( R_3 \)-holomorphic and never vanishes on \( R_3 \), there are no points on \( S \) at which \( R_3 = R_3 \), and therefore no umbilic points on \( S \). Thus, special \( R_3 \)-isothermal coordinates \( u, v \) in terms of which (6) is valid are locally available anywhere on \( S \). Moreover, for all such choices of \( u, v \),

\[
I < \frac{f + gk_2}{g(k_2 - k_1)}(dw^2 + dv^2) .
\]
On the other hand, the form
\[ \Gamma = (du^* + dv^2) \]
is well defined independently of the various choices of the special \( R_\Gamma \)-isothermal coordinates \( u, v \). Since \( I \) is complete, (10) implies that
\[ \frac{f + gk_z}{g(k_z - k_i)} \Gamma \]
is complete as well. Multiplication of a complete metric by a positive real valued function bounded away from zero will again yield a complete metric, so that if (9) is bounded from above, \( \Gamma' \) itself is complete on \( S \). Lifting the flat, complete, \( R_\Gamma \)-conformal metric \( \Gamma \) to the universal covering surface \( \Sigma \) of \( S \), we conclude that \( \Sigma \) is \( R_\Gamma \)-conformally the finite plane ([5], p. 394). But then, by the Remark following Lemma 3 above, the function (9) lifted to \( \Sigma \) is an \( R_\Gamma \)-subharmonic function bounded from above, and therefore must be constant. It follows finally from (4) that \( K = 0 \) on \( S \).

**COROLLARY.** \( H \) cannot be bounded away from zero on a complete, \( R'_2 \)-harmonically immersed surface \( S \) in \( E^3 \).

**Proof.** If \( S \) is \( R'_2 \)-harmonically immersed in \( E^3 \) with \( H \neq 0 \), then the \( R'_2 \)-holomorphic \( \Omega \) never vanishes. We apply Theorem 2 with
\[ f = \frac{-K}{\sqrt{H^2 - K}}, \quad g = \frac{H}{\sqrt{H^2 - K}} \]
so that
\[ \frac{k_z}{k_i + k_z} > 1 \]
is the expression (9). But (11) is bounded from above if and only if \( H \) is bounded away from zero. Thus, if \( H \) is bounded away from zero, \( K \equiv 0 \) on \( S \), a contradiction here since \( R'_2 \) is only defined where \( K < 0 \) on \( S \). In view of Lemma 6, we may reword the corollary to state that \( H \) cannot be bounded away from zero on a complete \( R'_2 \)-harmonically immersed surface in \( E^3 \).

It seems natural to ask whether there exists in \( E^3 \) a complete \( R'_2 \)-harmonically immersed \( S \) on which \( H \) never vanishes. Regardless of the answer to this question, our Corollary may have its best explanation in light of the Milnor conjecture ([4], p. 8) which contains the guess that \( H \) cannot be bounded away from zero on a complete surface in \( E^3 \) on which \( K \leq 0 \) unless \( K \equiv 0 \).
REFERENCES


Received December 14, 1965. This research was supported by grant NSF GP-2567 at UCLA and under a Ford Foundation grant at NYU.

UNIVERSITY OF CALIFORNIA, LOS ANGELES
<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friedrich-Wilhelm Bauer</td>
<td>Der Hurewicz-Satz</td>
<td>1</td>
</tr>
<tr>
<td>D. W. Dubois</td>
<td>A note on David Harrison’s theory of preprimes</td>
<td>15</td>
</tr>
<tr>
<td>Bert E. Fristedt</td>
<td>Sample function behavior of increasing processes with stationary, independent increments</td>
<td>21</td>
</tr>
<tr>
<td>Minoru Hasegawa</td>
<td>On the convergence of resolvents of operators</td>
<td>35</td>
</tr>
<tr>
<td>Søren Glud Johansen</td>
<td>The descriptive approach to the derivative of a set function with respect to a $\sigma$-lattice</td>
<td>49</td>
</tr>
<tr>
<td>John Frank Charles Kingman</td>
<td>Completely random measures</td>
<td>59</td>
</tr>
<tr>
<td>Tilla Weinstein</td>
<td>Surfaces harmonically immersed in $E^3$</td>
<td>79</td>
</tr>
<tr>
<td>Hikosaburo Komatsu</td>
<td>Fractional powers of operators. II. Interpolation spaces</td>
<td>89</td>
</tr>
<tr>
<td>Edward Milton Landesman</td>
<td>Hilbert-space methods in elliptic partial differential equations</td>
<td>113</td>
</tr>
<tr>
<td>O. Carruth McGehee</td>
<td>Certain isomorphisms between quotients of a group algebra</td>
<td>133</td>
</tr>
<tr>
<td>DeWayne Stanley Nymann</td>
<td>Dedekind groups</td>
<td>153</td>
</tr>
<tr>
<td>Sidney Charles Port</td>
<td>Hitting times for transient stable processes</td>
<td>161</td>
</tr>
<tr>
<td>Ralph Tyrrell Rockafellar</td>
<td>Duality and stability in extremum problems involving convex functions</td>
<td>167</td>
</tr>
<tr>
<td>Philip C. Tonne</td>
<td>Power-series and Hausdorff matrices</td>
<td>189</td>
</tr>
</tbody>
</table>