

# Pacific Journal of Mathematics

**HITTING TIMES FOR TRANSIENT STABLE PROCESSES**

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# HITTING TIMES FOR TRANSIENT STABLE PROCESSES

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**In this paper we explicitly find the asymptotic behavior, for large  $t$ , of the probability that a transient  $d$ -dimensional stable process first (last) hits a bounded Borel set during the time interval  $(t, \infty)$ .**

Assume that  $X(t)$  is a stable process on  $R^d$  ( $d$ -dimensional Euclidean space) having exponent  $\alpha < d$  and normalized so that the paths are right continuous with left-hand limits at every point. Assume further that  $[X(t) - X(0)]t^{-1/\alpha}$  is distributed like  $X(1) - X(0)$ , and moreover, that  $X(1) - X(0)$  has a genuinely  $d$ -dimensional distribution on  $R^d$ . [In particular, every symmetric stable process on  $R^d$  with 0 mean (when it exists) satisfies these conditions.]

From these assumptions it follows that  $X(t) - X(0)$  has a bounded, continuous density,  $f(t, x)$ , which satisfies the well-known scaling property

$$(1.1) \quad f(t, x) = t^{-d/\alpha} f(1, t^{-1/\alpha} x).$$

For a Borel (more generally, analytic) set  $B \subset R^d$ , let

$$V_B = \inf \{t > 0: X(t) \in B\}$$

denote the first hitting time of  $B$ . As usual we set  $V_B = \infty$  if

$$X(t) \notin B$$

for all  $t > 0$ . Our main purpose in this note is to establish the following.

**THEOREM 1.** *Let  $B$  be a bounded Borel (or analytic) subset of  $R^d$ . Then under the above assumptions on  $X(t)$ ,*

$$(1.2) \quad \lim_{t \rightarrow \infty} t^{(d/\alpha)-1} P_x(t < V_B < \infty) = P_x(V_B = \infty) C(B) \left[ \frac{d}{\alpha} - 1 \right]^{-1} f(1, 0),$$

where  $C(B)$  is the natural capacity of  $B$ .

Previously, (by using a different method) Joffe [2] established this result for symmetric processes with  $(d/2) < \alpha < 1$  when  $B$  has a non-empty interior, and Spitzer [4] (Lemma, p. 114) established this result for arbitrary compact  $B$  in the case of 3-dimensional Brownian motion. In the case of recurrent stable processes the analogue of Theorem 1 can be found in [3].

It is interesting to compare Theorem 1 with the following, much easier

**THEOREM 2.** *Let*

$$T_B = \inf\{t \geq 0: X(s) \notin B, \text{ all } s > t\}$$

*be the last hitting time of B. Then under the same conditions as Theorem 1,*

$$(1.3) \quad \lim_{t \rightarrow \infty} t^{(d/\alpha)-1} P_x(T_B > t) = C(B) \left[ \frac{d}{\alpha} - 1 \right]^{-1} f(1, 0) .$$

## 2 Proofs.

*Proof of Theorem 1.* A first passage decomposition yields

$$(2.1) \quad P_x(t < V_B < \infty) = \int_{R^d} P_x(V_B > t, X(t) \in dy) P_y(V_B < \infty) \\ = \int_{R^d} \left[ f(t, y - x) - \int_0^t \int_B H_B(x, ds, dz) f(t - s, y - z) \right] P_y(V_B < \infty) dy,$$

where here and in the following,

$$H_B(x, ds, dz) = P_x(V_B \in ds, X(s) \in dz) ,$$

and  $\bar{B}$  is the closure of  $B$ . But it is a known fact ([1] Prop. 18.4) that there is a measure,  $e_B(dy)$ , with support contained in  $\bar{B}$  (the capacity measure of  $B$ ) and finite total mass  $C(B)$  (the capacity of  $B$ ), such that

$$(2.2) \quad P_y(V_B < \infty) = \int_{\bar{B}} g(u - y) e_B(du) ,$$

where

$$g(x) = \int_0^\infty f(t, x) dt$$

is the potential kernel density for the process  $X(t)$ . Setting

$$R(t, x) = \int_t^\infty f(s, x) ds$$

and using the fact that

$$(2.3) \quad \int_{R^d} f(t, y - x) g(u - y) dy = R(t, u - x) ,$$

we obtain from (2.1) that

$$(2.4) \quad P_x(t < V_B < \infty) \\ \int_{\bar{B}} \left[ R(t, y - x) - \int_{\bar{B}} \int_0^t H_B(x, ds, dz) R(t - s, y - z) \right] e_B(dy) .$$

From the scaling property (1.1) and the fact that  $f(1, x)$  is continuous, we see that  $\lim_{t \rightarrow \infty} t^{d/\alpha} f(t, x) = f(1, 0)$ , uniformly in  $x$  on compacts, and thus

$$(2.5) \quad \lim_{t \rightarrow \infty} t^{(d/\alpha)-1} R(t, x) = f(1, 0) \left[ \frac{d}{\alpha} - 1 \right]^{-1} ,$$

uniformly in  $x$  on compacts. Set

$$R(t) = t^{-(d/\alpha)+1} \left[ \frac{d}{\alpha} - 1 \right]^{-1} .$$

Then from (2.5),

$$(2.6) \quad \lim_{t \rightarrow \infty} \int_{\bar{B}} \frac{R(t, y - x)}{R(t)} e_B(dy) = f(1, 0) C(B) ,$$

and

$$(2.7) \quad \lim_{T \rightarrow \infty} \lim_{t \rightarrow \infty} \int_0^T \left[ \int_{\bar{B}} \int_{\bar{B}} H_B(x, ds, dz) R(t - s, y - z) e_B(dy) \right] R(t)^{-1} \\ = \lim_{T \rightarrow \infty} \int_0^T H_B(x, ds, \bar{B}) C(B) f(1, 0) = P_x(V_B < \infty) C(B) f(1, 0) .$$

From (2.4), we see that in order to complete the proof it suffices to show

$$(2.8) \quad \lim_{T \rightarrow \infty} \limsup_{t \rightarrow \infty} R(t)^{-1} \int_T^t \int_{\bar{B}} H_B(x, ds, dz) R(t - s, y - z) e_B(dy) = 0 .$$

To accomplish this, decompose  $\int_T^t$  as  $\int_T^{t/2} + \int_{t/2}^{t-T} + \int_{t-T}^t$ . Since

$$\sup_x f(1, x) = K < \infty ,$$

it follows from the scaling property that  $R(t, x) \leq KR(t)$  for all  $t > 0$ . Setting  $A = KC(B)$ , we obtain

$$\int_T^{t/2} \leq A \int_T^{t/2} P_x(V_B \in ds) R(t - s) \leq AR(t/2) P_x(T < V_B < \infty) ,$$

and thus

$$\lim_T \limsup_t R(t)^{-1} \int_T^{t/2} = 0 .$$

Next observe that

$$\int_{t/2}^{t-T} \leq A \int_{t/2}^{t-T} P_x(V_B \in ds)R(t-s) \leq AR(T)P_x(t/2 < V_B < \infty) .$$

By (2.4) this last term is dominated by  $A^2R(T)R(t/2)$ , and thus

$$\lim_T \limsup_t R(t)^{-1} \int_{t/2}^{t-T} = 0 .$$

Finally, from (2.2) we see that

$$\int_{t-T}^t \leq \int_{t-T}^t \int_{\bar{B}} H_B(x, ds, dz) \int_{\bar{B}} g(y-z)e_B(dy) \leq \int_{t-T}^t P_x(V_B \in ds) .$$

But

$$\begin{aligned} P_x(t-T < V_B \leq t) &= \int_{R^d} P_x(V_B > t-T, X(t-T) \in dy)P_y(V_B \leq T) \\ &\leq \int_{R^d} f(t-T, y-x)P_y(V_B \leq T)dy \leq K(t-T)^{-d/\alpha} \int_{R^d} P_y(V_B \leq T)dy . \end{aligned}$$

Since the paths  $X(t)$  are bounded a.s. on  $[0, T]$ , we see that for each  $T$  there is a sphere  $S_T \supset \bar{B}$ , such that  $P_y(X(t) \in S_T) \geq 1/2$  for all  $t \leq T$  and  $y \in \bar{B}$ . But then

$$\begin{aligned} |S_T| &= \int_{R^d} P_x(X(T) \in S_T)dx \geq \int_{R^d} dx \int_0^T \int_{\bar{B}} H_B(x, ds, dy)P_y(X(T-s) \in S_T) \\ &\geq \frac{1}{2} \int_{R^d} P_x(V_B \leq T)dx . \end{aligned}$$

Thus

$$\lim_{t \rightarrow \infty} R(t)^{-1} \int_{t-T}^t = 0 .$$

This completes the proof.

*Proof of Theorem 2.* Clearly

$$P_x(T_B > t) = \int_{R^d} f(t, y-x)P_y(V_B < \infty)dy .$$

Using (2.2) and (2.3) we see that

$$P_x(T_B > t) = \int_{\bar{B}} R(t, y-x)e_B(dy) ,$$

from which the theorem follows.

REMARK. When  $d/2 < \alpha < d$ , it is possible to establish Theorem 1 by a much simpler argument. Set

$$Q_B^\lambda(x) = \int_0^\infty e^{-\lambda t} P_x(t < V_B < \infty) dt ,$$

$$H_B^\lambda(x, dy) = \int_0^\infty e^{-\lambda t} P_x(V_B \in dt, x(t) \in dy)$$

and

$$R^\lambda(x) = \int_0^\infty R(t, x) e^{-\lambda t} dt .$$

Then from (2.4) we obtain

$$(2.9) \quad Q_B^\lambda(x) = \int_{\bar{B}} \left[ R^\lambda(y - x) - \int_{\bar{B}} H_B^\lambda(x, dz) R^\lambda(y - z) \right] e_B(dy) .$$

It follows from (2.5) that uniformly in  $x$  on compacts,

$$\lim_{\lambda \downarrow 0} R^\lambda(x) \lambda^{2-d/\alpha} = f(1, 0) \left[ \frac{d}{\alpha} - 1 \right]^{-1} \Gamma(2 - d/\alpha) .$$

Consequently, from (2.9), we see that

$$\lim_{\lambda \downarrow 0} Q_B^\lambda(x) \lambda^{2-d/\alpha} = f(1, 0) C(B) P_x(V_B = \infty) \left[ \frac{d}{\alpha} - 1 \right]^{-1} \Gamma(2 - d/\alpha) .$$

An appeal to Karamata's theorem, and the fact that  $P_x(t < V_B < \infty)$  is monotone in  $t$ , then yields (1.2).

The above argument breaks down when  $\alpha < d/2$  since

$$\lim_{\lambda \downarrow 0} R^\lambda(x) < \infty ,$$

and the more complicated proof given previously is needed.

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