ON OPERATORS WHOSE FREDHOLM SET IS THE COMPLEX
PLANE

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Let $T$ be a closed linear operator with domain and range in a complex Banach space $X$. The Fredholm set $\Phi(T)$ of $T$ is the set of complex numbers $\lambda$ such that $\lambda - T$ is a Fredholm operator. If the space $X$ is of finite dimension then, obviously, the domain of $T$ is closed and $\Phi(T)$ is the whole complex plane $\mathbb{C}$. In this paper it is shown that the converse is also true. When $T$ is defined on all of $X$ this is a well-known result due to Gohberg and Krein.

Examples of nontrivial closed operators with $\Phi(T) = \mathbb{C}$ are the operators whose resolvent operator is compact. A characterization of the class of closed linear operators with a nonempty resolvent set and a Fredholm set equal to the complex plane will be given.

Throughout the present paper $X$ and $Y$ will denote complex Banach spaces. Let $T$ be an arbitrary closed linear operator with domain $\mathcal{D}(T)$ in $X$ and range $\mathcal{R}(T)$ in $Y$. The nullity $n(T)$ of $T$ is the dimension of the null space $\mathcal{N}(T)$ of $T$. The defect $d(T)$ of $T$ is the dimension of the quotient space $Y/\mathcal{R}(T)$. No distinction is made between infinite dimensions, so that $n(T)$ and $d(T)$ may be nonnegative integers or $+\infty$. We say that $T$ is Fredholm if $n(T)$ and $d(T)$ are both finite. Note that $d(T) < \infty$ implies $\mathcal{R}(T)$ is closed (cf. [5], Lemma 332).

In 1957 Gohberg and Krein [3] showed that if $A$ is a bounded linear operator on $X$ with $\Phi(A) = \mathbb{C}$, then the dimension of $X$ (denoted by $\dim X$) is finite. The following theorem extends this result.

**Theorem 1.** Let $T$ and $S$ be bounded linear operators from $X$ into $Y$. Suppose that $S$ is a homeomorphism, and that $T + \lambda S$ is Fredholm for each $\lambda \in \mathbb{C}$. Then

$$\dim X \leq \dim Y < +\infty.$$ 

**Proof.** Since $S$ is a homeomorphism, $\mathcal{R}(S)$ is closed and $n(S) = 0$. By a well-known stability theorem (cf. [5], Theorem 1), this implies the existence of a positive constant $\rho$ such that for $0 < |\mu| < \rho$

$$d(S) = d(S) - n(S) = d(S + \mu T) - n(S + \mu T).$$

The right-hand side is finite because $S + \mu T$ is Fredholm for $\mu \neq 0$. Hence $d(S) < +\infty$, and so $S$ has a bounded left inverse, say $R$. Then $n(R) \leq d(S) < +\infty$ and $d(R) = 0$, so $R$ is Fredholm. Define $A = RT$. 

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Then $A$ is a bounded linear operator on $X$ and
\[
\lambda - A = \lambda RS - RT = R(\lambda S - T).
\]
For each complex value of $\lambda$, $\lambda - A$ is the product of two bounded Fredholm operators and hence is Fredholm. But $\Phi(A) = C$ implies that $\dim X < \infty$ by the result of Gohberg and Kreĭn ([3], Theorem 3.2). Then $\dim Y = \dim X + d(S) < \infty$, concluding the proof.

**Corollary.** Let $T$ be a closed linear operator with domain $\mathcal{D}(T)$ and range in $X$. Then $\dim X < \infty$ if and only if $\mathcal{D}(T)$ is closed and $\Phi(T) = C$.

In [1] Caradus has proved that if $T$ is a closed linear operator with domain and range in $X$ such that $\dim X/\mathcal{D}(T) < \infty$, $\Phi(T) = C$ and such that the resolvent set of $T$ is neither empty nor the whole complex plane, then $\dim X < \infty$. The following lemma shows that Caradus’ result is contained in the Corollary.

**Lemma.** Let $T$ be a closed linear operator with domain in $X$ and range in $Y$. Suppose there exists a closed subspace $M$ of $X$ such that $X = \mathcal{D}(T) \oplus M$. Then $\mathcal{D}(T)$ is closed.

**Proof.** Let $Y_1$ be the Banach space $Y \times M$, with the norm
\[
\|(y, m)\| = \|y\| + \|m\|.
\]
Define the linear operator $J$ from $X$ into $Y_1$ by setting
\[
J(x + m) = (Tx, m)
\]
for each $x \in \mathcal{D}(T)$ and $m \in M$. It is easily verified that $J$ is a well-defined closed linear operator. Since the domain of $J$ is the Banach space $X$, the closed graph theorem implies that $J$ is bounded. Hence
\[
(\|Tx\| + \|m\|) \leq \|J\| \cdot \|x + m\|
\]
for each $x \in \mathcal{D}(T)$ and $m \in M$. In particular,
\[
\|Tx\| \leq \|J\| \cdot \|x\|
\]
for each $x \in \mathcal{D}(T)$. Thus $T$ is both closed and bounded, implying that $\mathcal{D}(T)$ is closed.

We have learned recently that similar statements for the range of a closed linear operator are proved by S. Goldberg in [4]. That this can be done follows easily from the observation that the range of a closed linear operator is always the domain of some other closed linear operator, and conversely (cf. [6], Chapter IV).
The Corollary states that the closed linear operators $T$ with closed domain and $\Phi(T) = C$ are trivial. Examples of nontrivial closed operators whose Fredholm set is the complex plane are the operators with compact resolvent (cf. [7], § 2). The following theorem shows that each closed operator $T$ with a nonempty resolvent set $\rho(T)$ and with $\Phi(T) = C$ is characterized by the fact that for each $\mu \in \rho(T)$ the resolvent $(\mu - T)^{-1}$ is a Riesz operator. For the definition of Riesz operators and one of their characterizations we refer to Dieudonné ([2], XI. 4, problem 5).

**Theorem 2.** Let $T$ be a closed linear operator with domain and range in $X$. If $\Phi(T) = C$, then $(\mu - T)^{-1}$ is a Riesz operator for all $\mu \in \rho(T)$. Conversely, if $(\mu - T)^{-1}$ is a Riesz operator for some $\mu \in \rho(T)$, then $\Phi(T) = C$.

**Proof.** We may assume that $\dim X = \infty$ and that $\rho(T)$ is not empty. Take $\mu$ in $\rho(T)$ and let $A = (\mu - T)^{-1}$. Then for $\lambda \neq \mu$,

$$(\lambda - T)(\mu - T)^{-1} = (\mu - \lambda)(\zeta - A),$$

where $\zeta = (\mu - \lambda)^{-1}$. This implies that $\Phi(T) = C$ if and only if $\Phi(A) = C \setminus \{0\}$. Hence it is enough to show that $A$ is a Riesz operator if and only if $\Phi(A) = C \setminus \{0\}$. In order to do this, let $\mathcal{H}$ be the ideal of all compact linear operators in the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on $X$, and let $\pi$ denote the canonical homomorphism from $\mathcal{L}(X)$ onto the quotient algebra $\mathcal{L}(X)/\mathcal{H}$. Then it follows from Atkinson's characterization of the class of all Fredholm operators in $\mathcal{L}(X)$ that $\zeta - A$ is Fredholm if and only if $\zeta - \pi(A)$ has an inverse in $\mathcal{L}(X)/\mathcal{H}$. So $\Phi(A) = C \setminus \{0\}$ if and only if the spectrum of $\pi(A)$ in $\mathcal{L}(X)/\mathcal{H}$ is $\{0\}$, i.e., the spectral radius $r(\pi(A))$ of $\pi(A)$ is zero. But

$$r(\pi(A)) = \lim_{n \to \infty} \| [\pi(A)]^n \|^{1/n} = \lim_{n \to \infty} \| \pi(A^n) \|^{1/n} = \lim_{n \to \infty} [d(A^n, \mathcal{H})]^{1/n},$$

where $d(A^n, \mathcal{H})$ is the infimum of $\| A^n - K \|$ for $K \in \mathcal{H}$. Thus $\Phi(A) = C \setminus \{0\}$ if and only if

$$\lim_{n \to \infty} [d(A^n, \mathcal{H})]^{1/n} = 0,$$

which is equivalent to the statement that $A$ is a Riesz operator (cf. [2], XI. 4, problem 5).

When $T$ is a self-adjoint closed linear operator in a Hilbert space Theorem 2 can be strengthened. This is because $(\mu - T)^{-1}$ is normal for $\mu \in \rho(T)$, and a normal operator is Riesz if and only if it is compact.
Hence, in this special case, $\Phi(T) = C$ if and only if $(\mu - T)^{-1}$ is compact for each $\mu$ in $\rho(T)$.

REFERENCES


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