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We shall prove that the homomorphisms and subalgebras of a multialgebra that can be studied naturally through its lattice-ordered representation as an ordinary algebra are limited to its ideal homomorphisms and Birkhoff subalgebras. However, these form a very limited subclass of the interesting homomorphisms and subalgebras. Of nearly equal importance, for example, are the co-ideal homomorphisms, which arise naturally in (say) groups from left coset decompositions by non-normal subgroups. To emphasize the special nature of ideal homomorphisms, the class of multiquasigroups is introduced, for which every regular mapping qualifies as a homomorphism. We show in general that ideal (co-ideal) homomorphisms correspond to equivalence relations which we call ideals (co-ideals), and the relationship between ideals, co-ideals, and coset decompositions in multiquasigroups is delineated.

The class of algebraic systems known as *multialgebras* [5], including its several specializations, e.g., multigroups [6, 7, 13, 14, 15], multirings [10], multilattices [1, 2], etc., seems eminently suited to the study of many problems. For example, *multigroups* have been used by Marty [9] to obtain results in the theory of rational functions of a complex variable, by Krasner [8] in class field theory, and by Prenowitz [13, 14, 15] in geometry. *Multilattices* have been used by Benado [1] to extend a result of Birkhoff [3, p. 149] relating to the relativistic wave equation. Nevertheless, there has been pointed out by several authors (Bruck [4], Dresher and Ore [7], Prenowitz [15]) a representation every multialgebra enjoys as a *lattice-ordered complete atomic algebra*, and Bruck [4, p. 42] has suggested that multialgebras are studied most appropriately from this point of view. Our intent is to prove, by theorem and example, that this is an inordinately modest conception of the richness of multivalued systems.

Indeed, we shall prove (Theorem 2) that the homomorphisms of a multialgebra that can be studied naturally through the induced homomorphisms of its lattice-ordered representation are limited to certain *ideal homomorphisms* corresponding to a class of equivalence relations on X which we call *ideals*.[1] We shall also show that if M

[1] The *ideals* and *co-ideals* introduced here are a specialization of a much broader concept occurring in the study of *relational algebras*. The concept of homomorphism introduced in § 2 is in fact inadequately discriminating, though it serves to justify the above thesis, and it is the author's intent to present in another paper a more thorough analysis of this concept, as it applies in relational algebras.

is an ordinary algebra, i.e., single valued, then its ideal homomorphs are ordinary algebras. But, one of the more interesting examples of a multigroup homomorphism is the mapping of a group, conceived of as multigroup, onto its left coset decomposition by a nonnormal subgroup. The homomorph thus obtained is always multivalued (cf., Dresher and Ore [7, p. 712]). This possibilty *properly* embeds the class of groups in the class of multigroups.

In Theorem 3 we prove a result about subalgebras similar to the stated result for homomorphisms. The subalgebras involved we call Birkhoff subalgebras. Benado [2] has invoked a much weaker concept. In § 4 we introduce the concept of *multiquasigroup* and obtain there certain results relating ideals, co-ideals, and coset decompositions. For example, we find that the equivalence relation determined by a left coset decomposition in a multiquasigroup satisfying the left associative law is a co-ideal. Theorem 9 generalizes a result of Birkhoff. § 5 contains several illustrative examples.

2. Homomorphisms of multialgebras. A *multialgebra* is a system $M = (X, f_1, f_2, \cdots)$, where each primitive operation f_i is a function *on* $X^{(k_i)}$ *to* the family of nonempty subsets of X. That is, given $x_1, \cdots, x_{k_i}$ in X, $f_i(x_1, \cdots, x_{k_i}) = A \subseteq X$, $A \neq \emptyset$. The sequence of integers $k_1, k_2, \cdots$ is the *species*, or *order type*, of the algebra. A multialgebra may also be viewed as a relational system with relations f_i of rank $k_i + 1$. A composition of these functions, say $f_i(f_j(x_1, \cdots, x_{k_j}), y_2, \cdots, y_{k_i})$, is defined to be the function whose values are

$$\bigcup_z f_i(z, y_2, \cdots, y_{k_i}) ,$$

where z runs over the elements of $f_j(x_1, \cdots, x_{k_j})$. There is a natural one-one correspondence between ordinary algebras, and those multialgebras in which every set A holds exactly one element.

In the sequel we shall denote a vector $(x_1, \cdots, x_k)$ by $\boldsymbol{x}$ and write $f(\boldsymbol{x})$ in place of $f(x_1, \cdots, x_k)$. We shall also adopt a useful notation of Prenowitz [14], and write $f(\boldsymbol{x}) \approx a$ (read $f(\boldsymbol{x})$ *meets* a) to mean $a \in f(\boldsymbol{x})$. This notation he extends to sets, writing $A \approx B$ to mean $A \cap B \neq \emptyset$. The relation is clearly reflexive and symmetric on the family of nonempty subsets of X, but not transitive. If h is a many-one map from X to X^* we let $h(A)$ denote $\{h(a) \mid a \in A\}$, and $h(\boldsymbol{x})$ denote $(h(x_1), \cdots, h(x_k))$ in the customary fashion. Also $h^{-1}(a^*) = \{a \mid h(a) = a^*\}$, and $h^{-1}(\boldsymbol{x}^*) = \{\boldsymbol{x} \mid h(\boldsymbol{x}) = \boldsymbol{x}^*\}$, where h^{-1} denotes the inverse of the map h.

A many-one map h from a multialgebra M onto a second multialgebra M^* of the same species is a (*regular*) *homomorphism* (Pickert [11]) if and only if the following two conditions are met

for each pair of corresponding primitive functions, f, f^*:

1. $f(\boldsymbol{x}) \approx a$ implies $f^*(h(\boldsymbol{x})) \approx h(a)$,
2. $f^*(\boldsymbol{x}^*) \approx a^*$ implies $f(h^{-1}(\boldsymbol{x}^*)) \approx h^{-1}(a^*)$.

A one-one map that is a homomorphism in both directions is an isomorphism. Condition 1 is equivalent to either of the following:

1′. $f^*(h(\boldsymbol{x})) \supseteq h(f(\boldsymbol{x}))$,

1″. $f(h^{-1}(\boldsymbol{x}^*)) \subseteq h^{-1}(f^*(\boldsymbol{x}^*))$.

If equality holds in either of 1′ or 1″ then 2 is satisfied. If equality holds in 1′ for every pair f, f^* we shall say h is an *ideal* homomorphism. This attractive generalization from the concept of homomorphism for ordinary algebras has been adopted by several authors (Bruck [4], Prenowitz [13]). If equality holds for 1″ for every pair f, f^* we shall say h is a *co-ideal* homomorphism. It can be shown that if $f(a_1, \cdots, a_n) \subseteq g(a_1, \cdots, a_n)$ is a compositional tautology in M, i.e., is valid for every substitution for $a_1, \cdots, a_n$, then the same compositional tautology is valid in both ideal and co-ideal homomorphs.

An equivalence relation E on X is an *ideal* (or ideal congruence relation) for M if and only if it satisfies for every primitive function f of M the condition: $f(\boldsymbol{x}) \approx a$ and $(\boldsymbol{x}, \boldsymbol{y}) \in E$ imply, for some b, that $f(\boldsymbol{y}) \approx b$ and $(a, b) \in E$. Here $(\boldsymbol{x}, \boldsymbol{y}) \in E$ means $(x_i, y_i) \in E$, $i = 1, \cdots, k$. E is a *co-ideal* for M if and only if it satisfies for every primitive function f the condition: $f(\boldsymbol{x}) \approx a$ and $(a, b) \in E$ imply, for some $\boldsymbol{y}$, that $f(\boldsymbol{y}) \approx b$ and $(\boldsymbol{x}, \boldsymbol{y}) \in E$. We note that if f is a single-valued function, everywhere defined, then the condition defining ideals is equivalent to the condition: $(\boldsymbol{x}, \boldsymbol{y}) \in E$ implies $(f(\boldsymbol{x}), f(\boldsymbol{y})) \in E$. Hence, if M is an ordinary algebra, E is an ideal if and only if it is a *congruence* relation for M (Birkhoff [3, p. vii]).

Corresponding to any equivalence relation E on X is a partition of X into pairwise disjoint, nonempty, subsets. We denote this family of subsets by X/E, and that member of the family to which an element x belongs by x/E. We also write A/E to denote $\{a/E \mid a \in A\}$ and $\boldsymbol{x}/E$ to denote $(x_1/E, \cdots, x_k/E)$. We define f/E to be that function f^* on $(X/E)^k$ to X/E such that $f^*(\boldsymbol{x}^*) = \bigcup \{A/E \mid$ for some $\boldsymbol{x}, f(\boldsymbol{x}) = A$ and $\boldsymbol{x}^* = \boldsymbol{x}/E\}$. Finally, $M/E = (X/E, f_1/E, f_2/E, \cdots)$ is the *factor algebra* determined by E. This is identical to the factor system obtained if M is regarded as a relational system (cf. Pickett [12]). If M is an ordinary algebra then M/E is an ordinary algebra if and only if E is a congruence relation. The map $p: x \to x/E$ is the *partition* map determined by E.

THEOREM 1. *If $h: M \to M^*$ is an ideal (co-ideal) homomorphism then $E = \{(x, y) \mid h(x) = h(y)\}$ is an ideal (co-ideal) of M. Conversely, if E is an ideal (co-ideal) of M then the partition map $p: M \to M/E$*

is an ideal (co-ideal) homomorphism. Moreover, the map $p(h^{-1}(x^*))$ *is an isomorphism between* M^* *and the factor algebra* M/E.

Proof. The verification that E is an equivalence relation is immediate. Now let f, f^* be corresponding functions of M and its ideal homomorph M^*, respectively. If $(\boldsymbol{x}, \boldsymbol{y}) \in E$ then $h(\boldsymbol{x}) = h(\boldsymbol{y})$ and hence

$$h(f(\boldsymbol{x})) = f^*(h(\boldsymbol{x})) = f^*(h(\boldsymbol{y})) = h(f(\boldsymbol{y})) \,.$$

One concludes that if $f(\boldsymbol{x}) \approx a$ and $(\boldsymbol{x}, \boldsymbol{y}) \in E$ then for some b, $f(\boldsymbol{y}) \approx b$ and $h(a) = h(b)$, i.e., $(a, b) \in E$. Therefore E is an ideal for M. For the converse let $f' = f/E$. If $f(\boldsymbol{x}) \approx a$ then $f'(\boldsymbol{x}/E) \approx a/E$ and hence $p(f(\boldsymbol{x})) \subseteqq f'(p(\boldsymbol{x}))$. Suppose $f'(p(\boldsymbol{x})) \approx b'$. Then for some $\boldsymbol{y}$, where $p(\boldsymbol{y}) = p(\boldsymbol{x})$, and some b, where $p(b) = b'$, we have $f(\boldsymbol{y}) \approx b$. Hence, $(\boldsymbol{x}, \boldsymbol{y}) \in E$, and $f(\boldsymbol{y}) \approx b$, and because E is an ideal, then for some a, $f(\boldsymbol{x}) \approx a$ and $(a, b) \in E$. Thus $p(a) = b'$, implying $f'(p(\boldsymbol{x})) \subseteqq p(f(\boldsymbol{x}))$. This proves p is an ideal homomorphism. The corresponding proofs for co-ideals are analogous, and the stated isomorphism is easily shown.

We now show how to construct the lattice ordered representation discussed in the introduction. Let $M = (X, f_1, f_2, \cdots)$ be a multialgebra, and let $\| X \|_0$ be the family of all nonempty subsets of X. Corresponding to each primitive function f_i of M we construct a single valued function F_i on $\| X \|_0$ in the following way. If $X_1, \cdots, X_{k_i}$ are in $\| X \|_0$ then $F_i(X_1, \cdots, X_{k_i}) = \bigcup f_i(x_1, \cdots, x_{k_i})$, where the union is over all $x_j \in X_j$, $j = 1, \cdots, k_i$. Under the ordering of set inclusion $\| X \|_0$ forms a partially ordered set, and if $X_1 \subseteqq Y_1, \cdots, X_{k_i} \subseteqq Y_{k_i}$, then

$$F_i(X_1, \cdots, X_{k_i}) \subseteqq F_i(Y_1, \cdots, Y_{k_i}) \,.$$

This makes the algebra $(\| X \|_0, F_1, F_2, \cdots)$ a partially ordered algebra, for which the set $\| X \|_0$ may be classified as a complete atomic lattice (ignoring the lack of a minimal element). Conversely, given such an algebra the space may be conceived as the set of nonempty subsets of some set X, and M may be retrieved by defining $f_i(x_1, \cdots, x_{k_i}) = F_i(\{x_1\}, \cdots, \{x_{k_i}\})$. Our discussion has followed Bruck [4, p. 42].

Suppose there is given a homomorphism h of a multialgebra $(X, f_1, f_2, \cdots)$ onto a multialgebra $(Y, g_1, g_2, \cdots)$ of the same species. Construct the algebras $(\| X \|_0, F_1, F_2, \cdots)$ and $(\| Y \|_0, G_1, G_2, \cdots)$ by the above method. The map h of X onto Y induces a map H of $\| X \|_0$ onto $\| Y \|_0$ defined by $H(A) = \{h(a) \mid a \in A\}$, for each A in $\| X \|_0$. G. Pickert [11, p. 379] has queried, under what conditions on h is the induced map H a homomorphism, i.e., an *ordinary* or algebraic homomorphism? We answer this with the following theorem.

THEOREM 2. *A necessary and sufficient condition that the induced map H be an ordinary homomorphism is that $E = \{(x, y) \mid h(x) = h(y)\}$ should be an ideal for $(X, f_1, f_2, \cdots)$.*

Proof. We shall use $\boldsymbol{X}$ to denote the vector $(X_1, \cdots, X_k)$, where $X_1, \cdots, X_k$ are in $\| X \|_0$. The induced map H determines an equivalence relation $\boldsymbol{E} = \{(A, B) \mid H(A) = H(B)\}$, and H is an algebraic homomorphism if and only if $\boldsymbol{E}$ is an algebraic congruence relation, that is, satisfies the condition

(I) $(\boldsymbol{X}, \boldsymbol{Y}) \in \boldsymbol{E}$ implies $(F(\boldsymbol{X}), F(\boldsymbol{Y})) \in \boldsymbol{E}$

for each primitive function F of $(\| X \|_0, F_1, F_2, \cdots)$. We note further that $(A, B) \in \boldsymbol{E}$ if and only if $a \in A$ implies, for some $b \in B$, that $(a, b) \in E$, and $b \in B$ implies, for some $a \in A$, that $(a, b) \in E$. Recall that E is an ideal for $(X, f_1, f_2, \cdots)$ if and only if it satisfies the condition

(II) $f(\boldsymbol{x}) \approx a$ and $(\boldsymbol{x}, \boldsymbol{y}) \in E$ imply, for some b, that $f(\boldsymbol{y}) \approx b$ and $(a, b) \in E$,

for each primitive function f. We must show that E is an ideal if and only if $\boldsymbol{E}$ is a congruence relation. (I) implies (II), for let $(\boldsymbol{x}, \boldsymbol{y}) \in E$ and $f(\boldsymbol{x}) \approx a$. Define $X_i = \{x_i\}$, $Y_i = \{y_i\}$, $i = 1, \cdots, k$. Then $(\boldsymbol{X}, \boldsymbol{Y}) \in \boldsymbol{E}$ and $F(\boldsymbol{X}) \approx a$. In view of condition (I), and the comment following, there exists b such that $F(\boldsymbol{Y}) \approx b$ and $(a, b) \in E$. Then $f(\boldsymbol{y}) \approx b$ and E is an ideal. (II) implies (I), for let $F(\boldsymbol{X}) \approx a$ and choose $x_1 \in X_1, \cdots, x_k \in X_k$ so that $f(\boldsymbol{x}) \approx a$. For a $\boldsymbol{Y}$ such that $(\boldsymbol{X}, \boldsymbol{Y}) \in \boldsymbol{E}$ choose $y_1 \in Y_1, \cdots, y_k \in Y_k$ so that $(\boldsymbol{x}, \boldsymbol{y}) \in E$. By II there exists b such that $f(\boldsymbol{y}) \approx b$ and $(a, b) \in E$. But then $F(\boldsymbol{Y}) \approx b$, so that $(F(\boldsymbol{X}), F(\boldsymbol{Y})) \in \boldsymbol{E}$, and hence $\boldsymbol{E}$ is a congruence relation.

In view of Theorem 1 this result may also be stated as follows: *H is an ordinary homomorphism if and only if h is an ideal homomorphism.* This leads to the following corollary. *If h is not an ideal homomorphism the factor algebra $(\| X \|_0, F_1, F_2, \cdots)/\boldsymbol{E}$ is multivalued.* For the only equivalence relations that preserve single-valuedness are the congruence relations. Hence, only by restricting one's attention to ideal homomorphism can multivalued systems be avoided. We shall try to indicate in §4 and §5 why this restriction is undesirable.

3. Submultialgebras. A *submultialgebra* of $(X, f_1, f_2, \cdots)$ is, for most of the generic types, understood to be a system $(Y, g_1, g_2, \cdots)$ of the same species, where $Y \subseteq X$, and for $y_1, \cdots, y_{k_i}$ in Y, $g_i(y_1, \cdots, y_{k_i}) = f_i(y_1, \cdots, y_{k_i}) \subseteq Y$. This seems to be a natural definition to employ, reflecting as it does Birkhoff's [3, p. vii] dictum that a subalgebra should "include every algebraic combination of its own elements". But, just as for ordinary algebras, one must not expect submulti-

algebras to possess all of the defining properties of the parent algebra.[2] We refer to these subsystems as *Birkhoff* subalgebras.

A weaker conception of subalgebra, sometimes of use, has been given by Benado [2, p. 313]. He defines $(Y, g_1, g_2, \cdots)$ to be a subalgebra if $g_i(y_1, \cdots, y_{k_i}) = Y \cap f_i(y_1, \cdots, y_{k_i}) \neq \varnothing$. It is clear that every Birkhoff subalgebra is a subalgebra under this definition. As in the case of ordinary algebras the family of all Birkhoff subalgebras is a lattice. With certain restrictions the subalgebras defined by Benado form a *multilattice*.

Suppose, now, that there is given a multialgebra $(X, f_1, f_2, \cdots)$, and a second multialgebra $(Y, g_1, g_2, \cdots)$, which is a subalgebra in the sense of Benado. Carry out the construction of the lattice-ordered representations of these systems, yielding $(\| X \|_0, F_1, F_2, \cdots)$ and $(\| Y \|_0, G_1, G_2, \cdots)$, respectively. Following the example of Pickert we ask, what restriction must be placed on the set Y in order that the second representation should be a subalgebra of the first? The answer is significant, but hardly surprising.

THEOREM 3. *A necessary and sufficient condition for* $(\| Y \|_0, G_1, G_2, \cdots)$ *to be a subalgebra of* $(\| X_0 \|, F_1, F_2, \cdots)$ *is that* $(Y, g_1, g_2, \cdots)$ *should be a Birkhoff subalgebra of* $(X, f_1, f_2, \cdots)$.

Proof. If $(\| Y \|_0, G_1, G_2, \cdots)$ is a subalgebra then, for each i, given $y_1, \cdots, y_{k_i}$ in Y,

$$\begin{aligned} f_i(y_1, \cdots, y_{k_i}) &= F_i(\{y_1\}, \cdots, \{y_{k_i}\}) \\ &= G_i(\{y_1\}, \cdots, \{y_{k_i}\}) \subseteqq Y , \end{aligned}$$

so $(Y, g_1, g_2, \cdots)$ is a Birkhoff subalgebra. Conversely, if $(Y, g_1, g_2, \cdots)$ is a Birkhoff subalgebra, then given $Y_1, \cdots, Y_{k_i}$ in $\| Y \|_0$,

$$\begin{aligned} G_i(Y_1, \cdots, Y_{k_i}) &= \bigcup g_i(y_1, \cdots, y_{k_i}) \\ &= \bigcup f_i(y_1, \cdots, y_{k_i}) = F_i(Y_1, \cdots, Y_{k_i}) , \end{aligned}$$

where the unions run over all $y_j \in Y_j$, $j = 1, \cdots, k_i$. Hence $(\| Y \|_0, G_1, G_2, \cdots)$ is a subalgebra.

We conclude that the only subsystems of multialgebras that can be studied naturally through the lattice-ordered representation are the Birkhoff subalgebras.

4. **Multiquasigroups.** A *multigroupoid* (Bruck [4]) $M = (X, \cdot)$ is a multialgebra with a single binary operation. We will call a multi-

[2] For example, the system $(Z, +)$ of integers under addition is a group, but the set of positive integers, though closed under all *additive* combinations, is only a semigroup.

groupoid a *multiquasigroup*[3] if and only if the conditions $a \cdot x \approx b$ and $y \cdot a \approx b$ have solutions for every a and b in X. A *multigroup* (Dresher and Ore [7]) is a multiquasigroup in which the associative law $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ holds. An element 1 of a multigroupoid is a *left unit* (*right unit*) if and only if $1 \cdot a \approx a$ ($a \cdot 1 \approx a$) for every a in X, and is a *unit* if and only if both conditions hold. A multiquasigroup with a unit is a *multiloop*. In § 5 we give a number of examples illustrating these systems.

The Birkhoff subalgebras of a multiquasigroup are not usually multiquasigroups. This may be overcome by introducing inverse operations $a \cdot / b = \{x \mid x \cdot b \approx a\}$ and $a / \cdot b = \{x \mid b \cdot x \approx a\}$. The Birkhoff submultialgebras of the algebra $(X, \cdot, \cdot/, /\cdot)$ correspond to certain submultiquasigroups of $(X, \cdot)$. These have been termed *closed* by Dresher and Ore, in the case of multigroups, because they contain with every a and b all solutions of $a \cdot x \approx b$ and $y \cdot a \approx b$.

According to our earlier definition a regular homomorphism of a multiquasigroup $M = (X, \cdot)$ onto a multiquasigroup $M^* = (X^*, \circ)$ is a many-one map h on X onto X^* satisfying

A1. $a \cdot b \approx c$ implies $h(a) \circ h(b) \approx h(c)$,

A2. $a^* \circ b^* \approx c^*$ implies $h^{-1}(a^*) \cdot h^{-1}(b^*) \approx h^{-1}(c^*)$.

We observe that M^* need not be specified to be a multiquasigroup, for given a^*, b^* belonging to M^* there are preimages a, b belonging to M, and also x and y belonging to M, such that $a \cdot x \approx b$, $y \cdot a \approx b$, so that $a^* \circ h(x) \approx b^*$, $h(y) \circ a^* \approx b^*$, and thus M^* is a multiquasigroup. Thus every regular map of M onto M^* deserves to be called a multiquasigroup homomorphism. Since there is a one-one correspondence (up to isomorphism) between the regular maps on M and the equivalence relations E on X, every factor system M/E should be called a homomorph of M.

Let $M = (X, \cdot)$ be a multiquasigroup, and Y a nonempty subset of X. A *left coset* of Y is a set $Y_a = Y \cdot a = \{b \mid$ for some $y \in Y, y \cdot a \approx b\}$. Y determines a *left coset decomposition* of X if and only if

B1. for each $a \in X$, $Y_a \approx a$,

B2. for each a, b in X, either $Y_a = Y_b$ or $Y_a \not\approx Y_b$.

The cosets of a left coset decomposition can be given a natural composition as follows. The product $Y_a \circ Y_b$ of two cosets is the set of all cosets Y_c such that for some a_1 in Y_a, for some b_1 in Y_b, $a_1 \cdot b_1 \approx Y_c$. Clearly the multiquasigroup $(\{Y_a, a \in X\}, \circ)$ is simply the factor algebra M/E, where E is the equivalence relation associated with the partition. Also clearly, the mapping $h: a \to Y_a$ is a regular homomorphism of M onto M/E. From our present point of view, coset decompositions are important because they provide examples of multialgebra homomor-

[3] Insofar as the author knows, such a concept has not previously been introduced.

phisms which are not determined by ideals. On the other hand, not all ideals are determined by coset decompositions. Examples to this effect are given in § 5.

If E is an equivalence relation on X let us write $a \equiv b(E)$, meaning $(a, b) \in E$. E is a *left congruence* relation for M if and only if there is a subset Y of X such that $a \equiv b(E)$ if and only if for some x in Y, $x \cdot a \approx b$.

THEOREM 4. *The left coset decompositions of a multiquasigroup M correspond one-one to the left congruence relations for M.*

Proof. Suppose first that Y determines a left coset decomposition of X. Define $a \equiv b(E)$ if and only if $Y_a \approx Y_b$, i.e., if and only if $Y_a = Y_b$. If $a \equiv b(E)$, then $b \approx Y_a$, so for some x in Y, $x \cdot a \approx b$. If for some x in Y, $x \cdot a \approx b$, then $Y_a = Y_b$ so $a \equiv b(E)$. Thus E is a left congruence relation. Next suppose E is a left congruence relation for M, and let Y be a subset of X with the specified property. Let $a \equiv b(E)$. Then for some x in Y, $x \cdot a \approx b$, so $b \approx Y_a$. Hence congruent elements belong to the same coset. Suppose $a \not\equiv b(E)$. Then for no x in Y does $x \cdot a \approx b$, so that $b \not\approx Y_a$. Thus non-congruent elements belong to different cosets.

It is of some interest to characterize those subsets Y giving rise to a left coset decomposition. We observe first that it is necessary and sufficient that Y satisfy, for every a, b, c in X, the following three conditions:

C1. $Y \cdot a \approx a$,

C2. $Y \cdot a \approx b$ implies $Y \cdot b \approx a$,

C3. $Y \cdot a \approx b$ and $Y \cdot b \approx c$ imply $Y \cdot a \approx c$.

If $Y \neq \varnothing$ we have for every a, for some b, that $Y \cdot a \approx b$. Then by C2 and C3 we obtain $Y \cdot a \approx a$. If Y is nonvoid a condition equivalent to C2 and C3 together is

C4. $Y \cdot a \approx b$ and $Y \cdot c \approx b$ imply $Y \cdot a \approx c$.

Hence if Y is nonvoid C4 characterizes Y. Observe that if Y_1 and Y_2 both determine the same left coset decomposition then $Y_1 \cup Y_2$ does also. This implies that to every left coset decomposition there corresponds a unique maximal set determining that decomposition. We call such a set *the* decomposition set of the decomposition. In general this set need not itself belong to the decomposition. If it does we call the decomposition *regular*.

The condition C3 is equivalent to the compositional relation $Y \cdot (Y \cdot a) \subseteq Y \cdot a$, for every a. If the *left associative* law $a \cdot (b \cdot c) \subseteq (a \cdot b) \cdot c$ is in effect in M, and if Y is any Birkhoff subalgebra, then the condition C3 is necessarily satisfied, for then

$$Y\cdot(Y\cdot a)\subseteq(Y\cdot Y)\cdot a\subseteq Y\cdot a\,.$$

In this case, to determine a left coset decomposition Y need only be a Birkhoff subalgebra satisfying C2. Dresher and Ore have, in the case of multigroups, called such subalgebras *left reversible.*

If Y determines a coset decomposition it is itself a coset if and only if $Y\cdot y=Y$, for every y in Y. It is therefore a Birkhoff subalgebra, and also has the property, which we call *left inversive,* that for every a, b in Y there is a solution in Y of $x\cdot a\approx b$. Conversely, a left inversive subalgebra Y satisfies $Y\cdot y=Y$, for every y in Y. In general, Y need not even be a Birkhoff subalgebra, and a necessary and sufficient condition for this is that $Y\cdot y\subseteq Y$ for every y in Y, i.e., the cosets Y_y must be contained in Y. In this case, these cosets are themselves Birkhoff subalgebras, for if a is in Y, then

$$Y_a\cdot Y_a\subseteq Y\cdot Y_a=Y_a\,.$$

However, if M satisfies the *right associative* law $(a\cdot b)\cdot c\subseteq a\cdot(b\cdot c)$, *the* decomposition set of a left coset decomposition is a Birkhoff subalgebra, for let Y be such a maximal set, and let $Z=Y\cdot Y\supseteq Y$. Then $Z\cdot a\approx b$ if and only if $Y\cdot a\approx b$, since clearly $Y\cdot a\approx b$ implies $Z\cdot a\approx b$, and if $Z\cdot a\approx b$, then $b\approx(Y\cdot Y)\cdot a\subseteq Y\cdot(Y\cdot a)=Y\cdot a$. Since Y is maximal, $Z=Y\cdot Y\subseteq Y$.

In order for a left coset decomposition to be regular the decomposition set must be a left inversive Birkhoff subalgebra. If a closed Birkhoff subalgebra Y determines a left coset decomposition, Y is the decomposition set, and the decomposition is regular. For clearly it is left inversive, and if $Z\cdot a=Y\cdot a$, for every a, then, for y in Y, $Z\cdot y\subseteq Y$, implying $Z\subseteq Y$. Again, if M has a right unit, and if a Birkhoff subalgebra Y determines a left coset decomposition, then Y is the decomposition set of a regular decomposition, for $Y\cdot 1\approx y, y$ in Y, implies $Y\cdot y\approx 1$, so 1 is in Y. Hence $Y\subseteq Y\cdot 1\subseteq Y$, i.e., $Y=Y\cdot 1$ is a coset. Therefore Y is left inversive. If $Z\cdot a=Y\cdot a$, for every a, then $Z\subseteq Z\cdot 1=Y\cdot 1=Y$, so Y is the maximal set determining the decomposition.

As a corollary to the above remarks we obtain the following result.

THEOREM 5. *Let M be a multigroup with a right unit. Then the left coset decompositions correspond one-one to the left reversible, left inversive, Birkhoff submultigroups, and all such decompositions are regular.*

In a similar fashion we can discuss right coset decompositions, and simultaneous left and right coset decompositions. If a set Y

determines left and right coset decompositions which are identical we shall simply refer to a coset decomposition. A necessary and sufficient condition for this to be true is that, for every a, $Y\cdot a = a\cdot Y$. Such subsets have been called *normal* by Dresher and Ore. A reversible Birkhoff subalgebra is a closed submultiquasigroup, since, e.g., $Y\cdot a \approx b \approx Y$ implies $Y\cdot b \approx a$, and thus a is in Y. Moreover, normality and the left conditions C1, C2, C3, or C4 imply the corresponding right conditions. We have the following theorem.

THEOREM 6. *The regular coset decompositions of a multiquasigroup M correspond one-one to the nonvoid normal Birkhoff subalgebras satisfying* C4. *These subalgebras are closed submultiquasigroups. If M satisfies either the left associative law or the right associative law the coset decomposition is always regular and corresponds one-one to its nonvoid normal reversible Birkhoff subalgebras.*

We turn now to the question of the relationship between ideals, co-ideals, and left coset decompositions. An ideal for a multigroupoid $M = (X, \cdot)$ is an equivalence relation E on X such that $a \equiv a_1(E)$ and $b \equiv b_1(E)$ and $a\cdot b \approx c$ imply, for some c_1, that $a_1\cdot b_1 \approx c_1$ and $c \equiv c_1(E)$. An equivalence relation E is a co-ideal for M if and only if $a\cdot b \approx c$ and $c \equiv c_1(E)$ imply, for some a_1, b_1, that $a \equiv a_1$ and $b \equiv b_1$ and $a_1\cdot b_1 \approx c_1$. In general there will be no left coset decomposition that coincides with X/E.

THEOREM 7. *Let Y determine a left coset decomposition of a multiquasigroup M and let E be the corresponding left congruence relation. Then*

a. *E is an ideal if and only if for every a, b, $(Y\cdot a)\cdot(Y\cdot b) \subseteqq Y\cdot(a\cdot b)$,*

b. *E is a co-ideal if and only if $(Y\cdot a)\cdot(Y\cdot b) \approx c$ implies $(Y\cdot a)\cdot(Y\cdot b) \supseteqq Y\cdot c$.*

Hence,

c. *E is both an ideal and a co-ideal if and only if $(Y\cdot a)\cdot(Y\cdot b) = Y\cdot(a\cdot b)$ for every a, b.*

d. *If E is an ideal then $Y\cdot Y$ is a Birkhoff subalgebra and $(Y\cdot Y)\cdot a \approx b$ if and only if $Y\cdot a \approx b$.*

Further, if M satisfies the left associative law then E is a co-ideal, and every compositional tautology that holds in M holds in M/E. If M is a multigroup and if Y is a normal Birkhoff subalgebra then E is an ideal.

Proof. Suppose E is an ideal. Let $a_1 \approx Y\cdot a$, $b_1 \approx Y\cdot b$, and $c_1 \approx$

$a_1 \cdot b_1$. Then $a \equiv a_1(E)$ and $b \equiv b_1(E)$ so that, for some $c \equiv c_1(E)$, $a \cdot b \approx c$, and $c_1 \approx Y \cdot c$. Hence $a_1 \cdot b_1 \subseteq Y \cdot (a \cdot b)$, implying $(Y \cdot a) \cdot (Y \cdot b) \subseteq Y \cdot (a \cdot b)$. Conversely, let $(Y \cdot a) \cdot (Y \cdot b) \subseteq Y \cdot (a \cdot b)$, for every a, b and let $a \equiv a_1(E)$, $b \equiv b_1(E)$. Then $(a \cdot b) \subseteq Y \cdot (a_1 \cdot b_1)$ and hence if $a \cdot b \approx c$ then for some $c_1 \approx a_1 \cdot b_1$, $c \approx Y \cdot c_1$, showing E is an ideal.

If E is a co-ideal, $(Y \cdot a) \cdot (Y \cdot b) \approx c$, and $c_1 \approx Y \cdot c$, then there exists $a_1 \equiv a(E)$, $b_1 \equiv b(E)$, such that $a_1 \cdot b_1 \approx c_1$. Hence $(Y \cdot a) \cdot (Y \cdot b) \supseteq Y \cdot c$. Conversely, if $(Y \cdot a) \cdot (Y \cdot b) \approx c$ implies $(Y \cdot a) \cdot (Y \cdot b) \supseteq Y \cdot c$, let $c_1 \equiv c(E)$ and $a \cdot b \approx c$. Then $(Y \cdot a) \cdot (Y \cdot b) \approx c$, so for some $a_1 \approx Y \cdot a$, some $b_1 \approx Y \cdot b$, $a_1 \cdot b_1 \approx c_1$. Hence E is a co-ideal. If M satisfies the left associative law, and if $(Y \cdot a) \cdot (Y \cdot b) \approx c$, then

$$(Y \cdot a) \cdot (Y \cdot b) = [Y \cdot (Y \cdot a)] \cdot (Y \cdot b) \supseteq Y \cdot [(Y \cdot a) \cdot (Y \cdot b)] \supseteq Y \cdot c \ ,$$

proving that E is necessarily a co-ideal.

If E is a co-ideal we see that $Y_c \approx Y_a \circ Y_b$ if and only if $c \approx (Y \cdot a) \cdot (Y \cdot b)$. Similarly, $Y_d \approx [Y_a \circ Y_b] \circ Y_c$ if and only if

$$d \approx [(Y \cdot a) \cdot (Y \cdot b)] \cdot (Y \cdot c) \ ,$$

since $Y_d \approx [Y_a \circ Y_b] \circ Y_c$ if and only if for some Y_e, $Y_e \approx Y_a \circ Y_b$ and $Y_d \approx Y_e \circ Y_c$, which is true if and only if for some e, $e \approx (Y \cdot a) \cdot (Y \cdot b)$ and $d \approx (Y \cdot e) \cdot (Y \cdot c)$ and hence if and only if $d \approx [(Y \cdot a) \cdot (Y \cdot b)] \cdot (Y \cdot c)$. Since every composition is built up from shorter compositions it is easy to give an inductive argument in the same way as above showing that, if f, f^* are corresponding compositions in $M, M/E$, then $Y_a \approx f^*(Y_{a_1}, \cdots, Y_{a_n})$ if and only if $a \approx f(Y \cdot a_1, \cdots, Y \cdot a_n)$. It immediately follows that if $f(a_1, \cdots, a_n) \subseteq g(a_1, \cdots, a_n)$ is a compositional tautology in M then $f^*(Y_{a_1}, \cdots, Y_{a_n}) \subseteq g^*(Y_{a_1}, \cdots, Y_{a_n})$ is a compositional tautology of M/E.

If M is a multigroup and Y is a normal Birkhoff subalgebra then

$$\begin{aligned}(Y \cdot a) \cdot (Y \cdot b) &= Y \cdot (a \cdot Y) \cdot b \\ &= Y \cdot (Y \cdot a) \cdot b = (Y \cdot Y) \cdot (a \cdot b) = Y \cdot (a \cdot b) \ ,\end{aligned}$$

and hence E is an ideal for M. If E is an ideal for a multiquasigroup M then $(Y \cdot Y) \cdot (Y \cdot Y) \subseteq Y \cdot (Y \cdot Y) = Y \cdot Y$, so that $Y \cdot Y$ is a Birkhoff subalgebra. Since $Y \cdot Y \supseteq Y$ it is clear that if $Y \cdot a \approx b$ then $(Y \cdot Y) \cdot a \approx b$. Suppose $(Y \cdot Y) \cdot a \approx b$. Then for some x, $x \cdot a \approx b$ and for some y, $Y \cdot y \approx x$. Since $a \equiv a(E)$ and $x \equiv y(E)$ and $x \cdot a \approx b$ there exists $c \equiv b$ such that $y \cdot a \approx c$. But then $a \equiv c \equiv b$, implying $Y \cdot a \approx b$. This completes the proof.

A further remark on the concept of compositional tautology may be clarifying. Although a relationship such as $(x \cdot x) \cdot y \subseteq x \cdot y$ may be a tautology in M, the left-hand side is not a composition in the above

sense. If the definition of composition is extended to allow identification of variables, as, e.g., $(x\cdot x)\cdot y$ is obtained from $(x\cdot u)\cdot y$ by identifying u and x, it is easy to construct examples in which coideals do not preserve all compositions. However, ideals preserve even this broader class of compositions.

Let E be an equivalence relation on X. If A and B are subsets of X we shall write $A \equiv B(E)$ if and only if for every x, $A \approx x/E$ if and only if $B \approx x/E$. This defines an equivalence relation on the subsets of X. Clearly, if E is an ideal for $M = (X, \cdot)$, and if $a \equiv a_1(E)$, $b \equiv b_1(E)$, also $a\cdot b \equiv a_1\cdot b_1(E)$. An element x of X such that, for every a of X, $x\cdot a \equiv a\cdot x \equiv a(E)$ we call a *congruent* of E.

THEOREM 8. *Let Y be the set of all congruents of an ideal E of a multiquasigroup $M = (X, \cdot)$. If $Y \neq \emptyset$, then*

a. *$(Y, \cdot)$ is a closed submultiquasigroup of M,*

b. *Y is an equivalence class of E,*

c. *M/E is a multiloop.*

Further, if E is a left congruence relation it is determined by Y, and M/E is a regular decomposition.

Proof. If x is in Y, and $y \equiv x$, then also y is in Y, for if $x\cdot a \equiv a\cdot x \equiv a$ then also $y\cdot a \equiv a\cdot y \equiv a$, since $y\cdot a \equiv x\cdot a$, $a\cdot y \equiv a\cdot x$. If x is in Y, y in Y, then $x \equiv y$, for $x \equiv y\cdot x \equiv x\cdot y \equiv y$. Hence Y is an equivalence class of E. If x, y are in Y and $z \approx x\cdot y \equiv x$ then $z \equiv x$ so z is in Y. If y, z are in Y and a solves $x\cdot y \approx z$ or $y\cdot x \approx z$ then a is in Y since $z \approx a\cdot y \equiv a$ or $z \approx y\cdot a \equiv a$ so $z \equiv a$. Therefore $(Y, \cdot)$ is a closed submultiquasigroup of M. If A is in X/E and y, a are in Y, A respectively, then if $b \approx y\cdot a \equiv a$, $b \equiv a$, so $Y \circ A \approx A$ in M/E. Similarly, $A \circ Y \approx A$ so Y is a unit for M/E. Suppose that E is a left congruence relation determined by Z, i.e., $a \equiv b(E)$ if and only if $Z\cdot a \approx b$. For z in Z, y in Y, if $z\cdot y \approx x$ then $y \equiv x \approx z\cdot y \equiv z$, showing z is in Y. Hence, if $a \equiv b(E)$ then $Z\cdot a \approx b$ implying $Y\cdot a \approx b$. On the other hand, if $Y\cdot a \approx b$ then for some y in Y, $y\cdot a \approx b$ implying $a \equiv y\cdot a \approx b$, so $a \equiv b$. Therefore E is determined by Y, and from the remarks preceding Theorem 5 it follows that M/E is a regular decomposition.

Finally, we shall generalize a theorem of Birkhoff [3, p. 86, Th. 2]. For D and E equivalence relations, $D \circ E = \{(x, z) \mid$ for some y, $(x, y) \in D$ and $(y, z) \in E\}$. If $D \circ E = E \circ D$, D and E are said to commute (or permute). Also, $D' = \{(y, x) \mid (x, y) \in D\}$.

THEOREM 9. *In a multiquasigroup a left congruence relation D commutes with every ideal E.*

Proof. Let Y be such that $a \equiv b(D)$ if and only if $Y \cdot a \approx b$. If $a \equiv x(D)$ and $x \equiv b(E)$ there exists u in Y such that $u \cdot x \approx a$. Since $u \equiv u(E)$ and $x \equiv b(E)$ and $u \cdot x \approx a$, there is some v such that $u \cdot b \approx v$ and $v \equiv a(E)$. Since u is in Y and $u \cdot b \approx v$, $b \equiv v(D)$, and hence $(a, b) \in E \circ D$. Therefore, $D \circ E \subseteq E \circ D = E' \circ D' = (D \circ E)'$. But for any binary relation A, if $A \subseteq A'$ then $A = A'$ since also $A' \subseteq (A')' = A$. Hence $D \circ E = E \circ D$, showing that D and E commute.

5. **Examples.** Multiquasigroups are extremely abundant. To form such a system on a set X it is only necessary to associate with each ordered pair (a, b) in X a nonvoid subset A of X in such a way that for every a, $\bigcup (a \cdot x) = X = \bigcup (x \cdot a)$, where x runs over the members of X. If X is finite such a system can be represented by an operation table in the usual fashion, the only novelty being the occurrence of multiple entries.

EXAMPLE 1. Let $X = \{e, a, b, c, d\}$ and define "$\cdot$" by the following table.

$\cdot$	e	a	b	c	d
e	e, a, b	e, a, b	e, a, b	c, d	c, d
a	a	b	e	c, d	e, a, b
b	b	e	a	e, a, b	c, d
c	e, c	a, c	b, c	c	d
d	e, d	a, d	b, d	d	c

TABLE 1.

It is easily checked that $(X, \cdot)$ is a multiquasigroup which satisfies neither the left nor the right associative law, but has a unit element, namely e. It is therefore a multiloop. It has a single left coset decomposition determined by $Y = \{e\}$, with cosets $Y \cdot e = Y \cdot a = Y \cdot b = \{e, a, b\}$, $Y \cdot c = Y \cdot d = \{c, d\}$. The left congruence relation is a co-ideal but not an ideal.

EXAMPLE 2. Let X be the set of complex numbers, and define $a \cdot b = \{x \mid x^2 + a^2 + b = 0\}$. This system is a noncommutative, nonassociative multiquasigroup with neither right nor left units.

Multiquasigroups also arise in the following way. Let $(X, \cdot)$ be a multiquasigroup. A new multiquasigroup $(X, \circ)$ is defined by setting $x \circ y = \{a \mid a \cdot x \approx y\}$. This system might justifiably be called the *left inverse* of $(X, \cdot)$. Naturally, all multigroups and all quasigroups are multiquasigroups. Below we give a few examples of multigroups. These are also quite prevalent.

EXAMPLE 3. Let $X_\alpha = \{\cos 0, \cos \pi/\alpha, \cos 2\pi/\alpha, \cdots\}$ and define

$$\cos m\pi/\alpha \cdot \cos n\pi/\alpha = \{\cos (m + n)\pi/\alpha, \cos (m - n)\pi/\alpha\} .$$

The commutative multigroup $(X_\alpha, \cdot)$ has $\cos 0$ as a unit. If α is an integer $(X_\alpha, \cdot)$ is a finite multigroup with $|\alpha| + 1$ elements. The operation table for $X_3 = \{1, 1/2, -1/2, -1\}$ is given in Table 2a. X_3 has a coset decomposition obtained by taking $Y = \{1, -1\}$, yielding $Y_1 = Y_{-1} = Y$ and $Y_{1/2} = Y_{-1/2} = \{1/2, -1/2\}$. The resulting homomorph is given in Table 2b. The corresponding equivalence relation is easily seen to be both an ideal and a co-ideal.

	1	1/2	−1/2	−1
1	1	1/2	−1/2	−1
1/2	1/2	1, −1/2	1/2, −1	−1/2
−1/2	−1/2	1/2, −1	1, −1/2	1/2
−1	−1	−1/2	1/2	1

TABLE 2a

	Y_1	$Y_{1/2}$
Y_1	Y_1	$Y_{1/2}$
$Y_{1/2}$	$Y_{1/2}$	$Y_1, Y_{1/2}$

TABLE 2b

EXAMPLE 4. Let $(X, \wedge, \vee, \leqq)$ be a lattice and define $a \cdot b = \{x \mid x \geqq a \wedge b\}$, $a \circ b = \{x \mid x \leqq a \vee b\}$. Both $(X, \cdot)$ and $(X, \circ)$ are commutative multigroups and every element is a unit. The only coset decomposition is determined by X, for if Y determined a coset decomposition for $(X, \cdot)$, say, then if a and b are any two elements of X, $Y \cdot (a \wedge b)$ meets both a and b. Hence they are in the same coset.

EXAMPLE 5. Let X be a nonvoid open convex subset of the plane, and define $a \cdot b = \{x \mid x$ lies on the open line segment joining $a, b\}$ if $a \neq b$, and define $a \cdot a = a$. $(X, \cdot)$ is a special case of a *join system* (Prenowitz [13]). In particular it is a commutative multigroup with no units. Many geometric concepts have algebraic analogs in this system. A convex subset A of X is a set such that if $x \in A$, $y \in A$ and z lies on the join of x and y then $z \in A$, i.e., $x \cdot y \subseteq A$. Thus a convex subset is a Birkhoff subalgebra. An open convex subset A also has the property that if $x \in A$, $y \in A$, then for some $z \in A$, y lies on the join of z and x, i.e., $y \approx z \cdot x$. Therefore an open convex subset is a submultigroup. A linear section L of X is subset holding, with each pair x, y in L, any point in X in the line on which they lie. L therefore holds $x \cdot y$, and all solutions to $a \cdot x \approx y$, $x \cdot b \approx y$. Hence L is a closed submultigroup. Conversely, any closed submultigroup is either X, a linear section, or $\varnothing$.

Because $b \cdot a \approx a$ in a join system only if $b = a$, if $Y \cdot a \approx a$ for every a in X then $Y = X$. Hence join systems have *no* proper left

coset decompositions. In the present example consider the decomposition of X consisting of a point a in X, and the open ray segments extending from a, lying in X. This is an example of an ideal, the resulting homomorph being a multigroup with a unit and unique inverses for every element. Geometrically it is analogous to a circle with its center point. One verifies easily that the set of congruents of this ideal is $\{a\} \neq \varnothing$.

EXAMPLE 6. Let $X = \{e, a, b\}$ and define $x \circ_1 y$, $x \circ_2 y$, and $x \circ_3 y$ by Tables 3, 4, 5, respectively. The multigroups $(X, \circ_1)$, $(X, \circ_2)$, $(X, \circ_3)$ are respectively commutative with no unit, commutative with a unit, and noncommutative with a unit. H. Campaigne [6] has shown that $(X, \circ_1)$ is not a homomorph of any group.

$\circ_1$	e	a	b
e	e	b	a
a	b	a	e, b
b	a	e, b	a, b

TABLE 3

$\circ_2$	e	a	b
e	e	a	b
a	a	a, b	e, b
b	b	e, b	a

TABLE 4

$\circ_3$	e	a	b
e	e	a	b
a	a	e, b	e, b
b	b	e, a	e, a

TABLE 5

EXAMPLE 7. Take X to be the dihedral group having generators a and b satisfying $a^4 = b^2 = e$, $ba = a^3b$. The subgroup $\{e, b\}$ determines the left coset decomposition $\{\{e, b\}, \{a, ab\}, \{a^2, a^2b\}, \{a^3, a^3b\}\}$. Under the operation defined in § 4 these cosets form a multigroup with the operation Table 6. The results of § 2 and § 4 show that the equivalence relation associated with the decomposition is a co-ideal but not an ideal.

	$\{e, b\}$	$\{a, ab\}$	$\{a^2, a^2b\}$	$\{a^3, a^3b\}$
$\{e, b\}$	$\{e, b\}$	$\{a, ab\}$, $\{a^3, a^3b\}$	$\{a^2, a^2b\}$	$\{a, ab\}$, $\{a^3, a^3b\}$
$\{a, ab\}$	$\{a, ab\}$	$\{e, b\}$, $\{a^2, a^2b\}$	$\{a^3, a^3b\}$	$\{e, b\}$, $\{a^2, a^2b\}$
$\{a^2, a^2b\}$	$\{a^2, a^2b\}$	$\{a, ab\}$, $\{a^3, a^3b\}$	$\{e, b\}$	$\{a, ab\}$, $\{a^3, a^3b\}$
$\{a^3, a^3b\}$	$\{a^3, a^3b\}$	$\{e, b\}$, $\{a^2, a^2b\}$	$\{a, ab\}$	$\{e, b\}$, $\{a^2, a^2b\}$

TABLE 6

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