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**CONCERNING NONNEGATIVE MATRICES AND DOUBLY
STOCHASTIC MATRICES**

RICHARD DENNIS SINKHORN AND PAUL JOSEPH KNOPP

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This paper is concerned with the condition for the convergence to a doubly stochastic limit of a sequence of matrices obtained from a nonnegative matrix A by alternately scaling the rows and columns of A and with the condition for the existence of diagonal matrices D_1 and D_2 with positive main diagonals such that D_1AD_2 is doubly stochastic.

The result is the following. The sequence of matrices converges to a doubly stochastic limit if and only if the matrix A contains at least one positive diagonal. A necessary and sufficient condition that there exist diagonal matrices D_1 and D_2 with positive main diagonals such that D_1AD_2 is both doubly stochastic and the limit of the iteration is that $A \neq 0$ and each positive entry of A is contained in a positive diagonal. The form D_1AD_2 is unique, and D_1 and D_2 are unique up to a positive scalar multiple if and only if A is fully indecomposable.

Sinkhorn [6] has shown that corresponding to each positive square matrix A there is a unique doubly stochastic matrix of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals. The matrices D_1 and D_2 are themselves unique up to a scalar factor. The matrix D_1AD_2 can be obtained as a limit of the sequence of matrices generated by alternately normalizing the rows and columns of A . But it was shown by example that for nonnegative matrices the iteration does not always converge, and even when it does, the D_1 and D_2 do not always exist.

Marcus and Newman [4] and Maxfield and Minc [5] gave some consideration to this problem for symmetric matrices.

In a recent communication with H. Schneider, the authors learned that Brualdi, Parter and Schneider [2] have independently obtained some of the results of this paper by employing different techniques.

DEFINITIONS. If A is an $N \times N$ matrix and σ is a permutation of $\{1, \dots, N\}$, then the sequence of elements $a_{1, \sigma(1)}, \dots, a_{N, \sigma(N)}$ is called the diagonal of A corresponding to σ . If σ is the identity, the diagonal is called the main diagonal.

If A is a nonnegative square matrix, A is said to have total support if $A \neq 0$ and if every positive element of A lies on a positive diagonal. A nonnegative matrix that contains a positive diagonal is said to have support.

The notation $A[\mu | \nu]$, $A(\mu | \nu)$, etc. is that of [3, pp. 10-11].

THEOREM. *Let A be a nonnegative $N \times N$ matrix. A necessary and sufficient condition that there exist a doubly stochastic matrix B of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals is that A has total support. If B exists then it is unique. Also D_1 and D_2 are unique up to a scalar multiple if and only if A is fully indecomposable.*

A necessary and sufficient condition that the iterative process of alternately normalizing the rows and columns of A will converge to a doubly stochastic limit is that A has support. If A has total support, this limit is the described matrix D_1AD_2 . If A has support which is not total, this limit cannot be of the form D_1AD_2 .

Proof. We first demonstrate uniqueness. Suppose $B = D_1AD_2$ and $B' = D'_1AD'_2$ are doubly stochastic where $D_1 = \text{diag}(x_1, \dots, x_N)$, $D_2 = \text{diag}(y_1, \dots, y_N)$, $D'_1 = \text{diag}(x'_1, \dots, x'_N)$, and $D'_2 = \text{diag}(y'_1, \dots, y'_N)$. If $p_i = x'_i/x_i$, $q_j = y'_j/y_j$,

$$\begin{aligned} \sum_i x_i a_{ij} y_j &= 1; & \sum_j x_i a_{ij} y_j &= 1 \\ \sum_i p_i x_i a_{ij} q_j y_j &= 1; & \sum_j p_i x_i a_{ij} q_j y_j &= 1. \end{aligned}$$

Let $E_j = \{i \mid a_{ij} > 0\}$, $F_i = \{j \mid a_{ij} > 0\}$ and put

$$m = \{i \mid p_i = \min_i p_i = \underline{p}\}, M = \{j \mid q_j = \max_j q_j = \bar{q}\}.$$

Pick $i_0 \in m$, $j_0 \in M$. Then $q_{j_0} = (\sum_i p_i x_i a_{ij_0} y_{j_0})^{-1} \leq p_{i_0}^{-1}$ and similarly $p_{i_0} \geq q_{j_0}^{-1}$, forcing $q_{j_0} = p_{i_0}^{-1} = \underline{p}^{-1}$. But equality is possible only if $p_i = \underline{p}$ when $i \in E_{j_0}$. Whence $p_i = \underline{p}$ when $i \in E_j$ and $j \in M$. Thus $\bigcup_{j \in M} E_j \subseteq m$ and it follows that $A(m \mid M) = 0$. In the same way $p_{i_0} = q_{j_0}^{-1}$ is possible only if $q_j = \bar{q}$ for all $j \in F_{i_0}$. Whence $q_j = \bar{q}$ when $j \in F_i$ and $i \in m$. Thus $\bigcup_{i \in m} F_i \subseteq M$ and it follows that $A[m \mid M] = 0$.

On $m \times M$, $p_i q_j = \underline{p} \bar{q} = 1$ and it follows that $B[m \mid M] = B'[m \mid M]$ is doubly stochastic. In particular m and M must have the same size.

If A is fully indecomposable, $A(m \mid M)$ and $A[m \mid M]$ thus cannot exist. In such a case $A = A[m \mid M]$. Thus $D_1AD_2 = D'_1AD'_2$, and D_1 and D_2 are themselves unique up to a scalar multiple.

If $A(m \mid M)$ and $A[m \mid M]$ exist, $B(m \mid M)$ and $B'(m \mid M)$ exist and are each doubly stochastic matrices of order less than N . Furthermore $B(m \mid M) = D''_1 A(m \mid M) D''_2$ and $B'(m \mid M) = D'''_1 A(m \mid M) D'''_2$ where the D 's are diagonal matrices with positive main diagonals. The argument may be repeated on these submatrices until $D_1AD_2 = D'_1AD'_2$ is established. Note that D_1 and D_2 no longer need be unique up to a scalar multiple.

The necessity of total support for the existence of D_1AD_2 is an immediate consequence of the celebrated theorem of G. Birkhoff [1] which states that the set of doubly stochastic matrices of order N is the convex hull of the $N \times N$ permutation matrices.

The sufficiency of the condition and the remarks concerning the iteration will follow in part from the following lemmas.

LEMMA 1. *If A is a row stochastic matrix, and β_1, \dots, β_N are the column sums of A , then $\prod_{k=1}^N \beta_k \leq 1$, with equality only if each $\beta_k = 1$.*

Proof. Let A have column sums β_1, \dots, β_N . Of course, each $\beta_k \geq 0$ and $\sum_{k=1}^N \beta_k = N$. By the arithmetic-geometric mean inequality

$$\prod_{k=1}^N \beta_k \leq \left[(1/N) \sum_{k=1}^N \beta_k \right]^N = 1$$

with equality occurring only if each $\beta_k = 1$.

LEMMA 2. *Let $A = (a_{ij})$ be an $N \times N$ matrix with total support and suppose that if $1 \leq i, j \leq N$, $\{x_{i,n}\}$ and $\{y_{j,n}\}$ are positive sequences such that $x_{i,n}y_{j,n}$ converges to a positive limit for each i, j such that $a_{ij} \neq 0$. Then there exist convergent positive sequences $\{x'_{i,n}\}, \{y'_{j,n}\}$ with positive limits such that $x'_{i,n}y'_{j,n} = x_{i,n}y_{j,n}$ for all i, j, n .*

Proof. Consider first the case in which A is fully indecomposable. Let $E^{(1)} = \{1\}$, $F^{(1)} = \{j \mid a_{1j} > 0\}$, and $E^{(s)} = \{i \notin \bigcup_{k=1}^{s-1} E^{(k)} \mid \text{for some } j \in F^{(s-1)}, a_{ij} > 0\}$, $F^{(s)} = \{j \notin \bigcup_{k=1}^{s-1} F^{(k)} \mid \text{for some } i \in E^{(s)}, a_{ij} > 0\}$ when $s > 1$. The sets $E^{(s)}$ and $F^{(s)}$ are void for sufficiently large s , e.g., for $s > N$. Define $E = \bigcup_k E^{(k)}$ and $F = \bigcup_k F^{(k)}$. Since A has total support, the first row of A contains a nonzero element; thus $F^{(1)}$ is nonvoid. Since $F^{(1)} \subseteq F$, F is nonvoid. Also since $\{1\} = E^{(1)} \subseteq E$, E is nonvoid.

Suppose E is a proper subset of $\{1, \dots, N\}$. Pick $i \notin E, j \in F$. Then $j \in F^{(s)}$ for some s . Since $i \notin E$, certainly $i \notin \bigcup_{k=1}^s E^{(k)}$. Certainly then it could not be that $a_{ij} > 0$ for then $i \in E^{(s+1)} \subseteq E$, a contradiction. Whence $i \notin E, j \in F \Rightarrow a_{ij} = 0$, i.e., $A(E|F) = 0$. In the same way it follows that if $F \neq \{1, \dots, N\}$, $A[E|F) = 0$.

Define an $N \times N$ matrix $H = (h_{ij})$ as follows. If $a_{ij} = 0$, set $h_{ij} = 0$. If $a_{ij} \neq 0$ and a_{ij} lies on t positive diagonals in A , set $h_{ij} = t/\tau$ where τ is the total number of positive diagonals in A . Then H is doubly stochastic and $h_{ij} = 0$ if and only if $a_{ij} = 0$. Suppose E contains u elements and F contains v elements. Since $H(E|F) = 0$, $\sum_{i \in E} \sum_{j \in F} h_{ij} = v$, and since either $F = \{1, \dots, N\}$ or $H[E|F) = 0$, $\sum_{i \in E} \sum_{j \in F} h_{ij} = u$. Thus E and F have the same number of elements.

But E and F cannot be proper subsets of $\{1, \dots, N\}$ if A is assumed to be fully indecomposable. Thus $E = F = \{1, \dots, N\}$.

Define $x'_{i,n} = x_{1,n}^{-1}x_{i,n}$ and $y'_{j,n} = x_{1,n}y_{j,n}$ for all i, j, n . Then $x'_{i,n}y'_{j,n} = x_{i,n}y_{j,n}$ for all i, j, n . Since $x'_{1,n} = 1$ for all n , certainly $x'_{i,n} \rightarrow 1$. For $j \in F^{(1)}$, $y'_{j,n} = x'_{1,n}y'_{j,n} = x_{1,n}y_{j,n}$ has a positive limit.

Inductively suppose that it is known that $x'_{i,n}$ and $y'_{j,n}$ converge to positive limits when $i \in \bigcup_{k=1}^{s-1} E^{(k)}$ and $j \in \bigcup_{k=1}^{s-1} F^{(k)}$. For $i \in E^{(s)}$ there is a $j_{s-1} \in F^{(s-1)}$ such that $a_{ij_{s-1}} > 0$. Thus $x'_{i,n} = x'_{i,n}y'_{j_{s-1},n}/y'_{j_{s-1},n} = x_{i,n}y_{j_{s-1},n}/y'_{j_{s-1},n}$ has a positive limit. Then for $j \in F^{(s)}$ there is a $i_s \in E^{(s)}$ such that $a_{i_s j} > 0$. Whence $y'_{j,n} = x'_{i_s,n}y'_{i_s,n}/x'_{i_s,n} = x_{i_s,n}y_{j,n}/x'_{i_s,n}$ has a positive limit. This completes the proof in case A is fully indecomposable.

If A is not fully indecomposable, then neither is the corresponding doubly stochastic matrix H . This means that there exist permutations P and Q such that $PHQ = H_1 \oplus \dots \oplus H_g$ where each H_k is doubly stochastic and fully indecomposable. Thus also $PAQ = A_1 \oplus \dots \oplus A_g$ where each A_k has total support and is fully indecomposable. The above argument may be repeated on each of the A_k .

Now we return to the theorem. Suppose A has support. Define an iteration on A as follows.

Let $x_{i,0} \equiv 1$, $y_{j,0} \equiv (\sum_{i=1}^N a_{ij})^{-1}$ and set $x_{i,n+1} = \alpha_{i,n}^{-1}x_{i,n}$, $y_{j,n+1} = \beta_{j,n}^{-1}y_{j,n}$ where

$$\alpha_{i,n} = \sum_{j=1}^N x_{i,n}a_{ij}y_{j,n}; \quad \beta_{j,n} = \sum_{i=1}^N \alpha_{i,n}^{-1}x_{i,n}a_{ij}y_{j,n},$$

$i = 1, \dots, N, j = 1, \dots, N, n = 0, 1, \dots$. Note that $(x_{i,n}a_{ij}y_{j,n})$ is column stochastic and $(x_{i,n+1}a_{ij}y_{j,n})$ is row stochastic. Then in particular

$$y_{j,n} = \left(\sum_{i=1}^N x_{i,n}a_{ij} \right)^{-1} \leq x_{i_0,n}^{-1}a_{i_0 j}^{-1} \leq x_{i_0,n}^{-1}a^{-1}$$

where i_0 is such that $a_{i_0 j} > 0$ and a is the minimal positive a_{ij} . Thus $x_{i,n}y_{j,n} \leq a^{-1}$ if $a_{ij} > 0$.

Let A have a positive diagonal corresponding to a permutation σ , and set $s_n = \prod_{i=1}^N x_{i,n}y_{\sigma(i),n}$ and $s'_n = \prod_{i=1}^N x_{i,n+1}y_{\sigma(i),n}$. By Lemma 1 and the preceding remark, $s_n \leq s'_n \leq s_{n+1} \leq a^{-N}$. Thus $s_n \rightarrow L$ and $s'_n \rightarrow L$ where $0 < L \leq a^{-N}$. Whence $\prod_{j=1}^N \beta_{j,n} = s'_n/s_{n+1} \rightarrow 1$. This is impossible unless each $\beta_{j,n} \rightarrow 1$ since $\prod_{k=1}^N \beta_k$ has a unique maximal value of 1 only when $\beta_1 = \dots = \beta_N = 1$. Similarly each $\alpha_{i,n} \rightarrow 1$.

Thus if A has a positive diagonal, the limit points of the sequence of matrices generated by the iteration are doubly stochastic. However, two such limit points are diagonally equivalent. Suppose that A_n is the n th matrix in the iteration and that $A_{n_k} \rightarrow B$ and $A_{m_k} \rightarrow C$. Observe that for any given pair i, j $b_{ij} \neq 0 \Leftrightarrow c_{ij} \neq 0$. For any per-

mutation σ , $\prod_{i=1}^N b_{i,\sigma(i)} = \prod_{i=1}^N c_{i,\sigma(i)} = L \prod_{i=1}^N a_{i,\sigma(i)}$. Then certainly $b_{ij} \neq 0 \Rightarrow c_{ij} \neq 0$, for suppose $b_{i_0j_0} \neq 0$. Then $b_{i_0j_0}$ lies on a positive diagonal. The corresponding diagonal in C would have a positive product. Thus $c_{i_0j_0} \neq 0$. In the same way $c_{ij} \neq 0 \Rightarrow b_{ij} \neq 0$. If in addition A has total support then $a_{ij} \neq 0 \Leftrightarrow b_{ij} \neq 0 \Leftrightarrow c_{ij} \neq 0$.

By construction there exist matrices $\tilde{D}_{1,k} = \text{diag}(w_{1,k}, \dots, w_{N,k})$ and $\tilde{D}_{2,k} = \text{diag}(z_{1,k}, \dots, z_{N,k})$ with positive main diagonals such that $A_{m_k} = \tilde{D}_{1,k} A_{n_k} \tilde{D}_{2,k}$. For $b_{ij} > 0$, $w_{i,k} z_{j,k} \rightarrow c_{ij} b_{ij}^{-1}$. By Lemma 2 there exist positive sequences $\{w'_{i,k}\}$ and $\{z'_{j,k}\}$ converging to positive limits such that $w'_{i,k} z'_{j,k} = w_{i,k} z_{j,k}$, for all i, j, k . If

$$D_1 = \lim_{k \rightarrow \infty} \text{diag}(w'_{1,k}, \dots, w'_{N,k}) \quad \text{and} \quad D_2 = \lim_{k \rightarrow \infty} \text{diag}(z'_{1,k}, \dots, z'_{N,k}),$$

then $C = D_1 B D_2$. By the uniqueness part of the theorem, $B = C$. It follows that the iteration converges. It is clear from Birkhoff's theorem that no limit to the iteration is possible without at least one positive diagonal.

Suppose A has total support. Let $D_{1,n} = \text{diag}(x_{1,n}, \dots, x_{N,n})$ and $D_{2,n} = \text{diag}(y_{1,n}, \dots, y_{N,n})$. Then $B = \lim_{n \rightarrow \infty} D_{1,n} A D_{2,n}$ exists and $b_{ij} \neq 0 \Leftrightarrow a_{ij} \neq 0$. When $a_{ij} > 0$, $x_{i,n} y_{j,n} \rightarrow b_{ij} a_{ij}^{-1}$. By Lemma 2 there are convergent positive sequences $\{x'_{i,n}\}$, $\{y'_{j,n}\}$ with positive limits such that $x'_{i,n} y'_{j,n} = x_{i,n} y_{j,n}$ for all i, j, n . Let $D_1 = \lim_{n \rightarrow \infty} \text{diag}(x'_{1,n}, \dots, x'_{N,n})$ and $D_2 = \lim_{n \rightarrow \infty} \text{diag}(y'_{1,n}, \dots, y'_{N,n})$. Then $B = D_1 A D_2$.

Finally we observe that if A has support which is not total, then by Birkhoff's theorem, there is a nonzero element of A which tends to zero in the iteration. In fact every nonzero element of A which is not on a positive diagonal must do so. If the limit matrix could be put in the form $D_1 A D_2$ then some term $x_i a_{ij} y_j = 0$ where $a_{ij} > 0$. But then either $x_i = 0$ or $y_j = 0$. The former leads to a row of zeros and the latter to a column of zeros in $D_1 A D_2$. In either case $D_1 A D_2$ could not be doubly stochastic.

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