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In this paper, the authors define the order sum of a family of distributive lattices which is indexed by a partially ordered set P. The order sum reduces to the free product when P is trivially ordered, and to the ordinal sum when P is simply ordered.

It is proved that the order sum of conditionally implicative lattices is conditionally implicative, and that every projective distributive lattice is conditionally implicative. The second half of the paper investigates conditions under which the order sum of projective lattices is projective. It is shown that if $\{L_\alpha \mid \alpha \in P\}$ is a family of distributive lattices having largest and smallest elements, then the order sum of the family is projective if and only if each L_α is projective, and P is such that the order sum of the family $\{M_\alpha \mid \alpha \in P\}$ of one-element lattices M_α is projective.

1. **Existence and properties of order sums.**

DEFINITION 1.1. Let $\{L_\alpha \mid \alpha \in P\}$ be a family of distributive lattices whose index set P is partially ordered. A distributive lattice L is called the *order sum* of the family if for each $\alpha \in P$ there exists a monomorphism $\varphi_\alpha: L_\alpha \to L$ such that:

(1.1) L is generated by $\bigcup_{\alpha \in P} \varphi_\alpha(L_\alpha)$.

(1.2)   If $\alpha < \beta$ and $x \in L_\alpha$, $y \in L_\beta$, then $\varphi_\alpha(x) < \varphi_\beta(y)$.

(1.3)   If M is any distributive lattice and $\{f_\alpha: L_\alpha \to M \mid \alpha \in P\}$ is a family of homomorphisms such that whenever $\alpha < \beta$, $f_\alpha(x) \leqq f_\beta(y)$ for all $x \in L_\alpha$ and $y \in L_\beta$, then there exists a homomorphism $f: L \to M$ such that $f\varphi_\alpha = f_\alpha$ for all $\alpha \in P$.

The existence of the order sum will be proved in Theorem 1.7.

THEOREM 1.2. *If L and M are order sums of $\{L_\alpha \mid \alpha \in P\}$, then $L \cong M$.*

Proof. Let $\varphi_\alpha: L_\alpha \to L$ and $\psi_\alpha: L_\alpha \to M$ be monomorphisms as in Definition 1.1. By (1.3), there exist homomorphisms $f: M \to L$ and $g: L \to M$ such that $f\psi_\alpha = \varphi_\alpha$ and $g\varphi_\alpha = \psi_\alpha$. We have $fg\varphi_\alpha = f\psi_\alpha = \varphi_\alpha$ for each α, and therefore by (1.1), $fg = I_L$, the identity function on L. Similarly $gf = I_M$. Thus $g = f^{-1}$ and f is an isomorphism.

By Theorem 1.2, the order sum of a family $\{L_\alpha \mid \alpha \in P\}$ is unique up to isomorphism. We denote the order sum by $\Sigma_{\alpha \in P} L_\alpha$. If P is

trivially ordered, in the sense that $\alpha \leq \beta$ only when $\alpha = \beta$, then the order sum reduces by very definition to the free product [2].

DEFINITION 1.3. Let $\{L_\alpha \mid \alpha \in P\}$ be a family of pairwise disjoint lattices indexed by a chain P. The *ordinal sum* L of the family is the set $\bigcup_{\alpha \in P} L_\alpha$ with the following partial order: if $x \in L_\alpha$ and $y \in L_\beta$, then $x \leq y$ if and only if either $\alpha < \beta$, or $\alpha = \beta$ and $x \leq y$ in the original order of L_α. If the lattices L_α are not pairwise disjoint, then the ordinal sum of the L_α is defined to be the ordinal sum of pairwise disjoint lattices $\tilde{L}_\alpha$ such that $\tilde{L}_\alpha \cong L_\alpha$. If P is the chain $\{0, 1\}$ with $0 < 1$, then the ordinal sum of $\{L_0, L_1\}$ is denoted by $L_0 \oplus L_1$.

THEOREM 1.4. *If P is a chain, then the ordinal sum L of $\{L_\alpha \mid \alpha \in P\}$ is the order sum $\Sigma_{\alpha \in P} L_\alpha$.*

Proof. We may assume the L_α are pairwise disjoint. Let $\varphi_\alpha : L_\alpha \to L$ be the inclusion map. Obviously conditions (1.1) and (1.2) hold. Suppose the f_α are as in (1.3). Let $f : L \to M$ be the union (combined function) of the f_α. It is clear that f is a homomorphism and $f \varphi_\alpha = f_\alpha$.

COROLLARY 1.5. *If P is a chain, and for each $\alpha \in P, L_\alpha$ is a one-element lattice, then $\Sigma_{\alpha \in P} L_\alpha \cong P$.*

Proof. In this case the ordinal sum of the L_α is obviously isomorphic with P.

THEOREM 1.6. *Let $\{x_i \mid i \in I\}$ be a family of distinct variables. Let E be a set of inequalities of the form $x_{i_1} \cdot \dots \cdot x_{i_m} \leq x_{j_1} + \dots + x_{j_n}$, where the indices are in I. Then there exists a distributive lattice L which is generated by a family $\{a_i \mid i \in I\}$ of elements such that*
    (1.4) *$\{a_i \mid i \in I\}$ satisfies every inequality in E.*
    (1.5) *If M is a distributive lattice which contains a family $\{b_i \mid i \in I\}$ which satisfies every inequality in E, then there exists a homomorphism $f : L \to M$ such that $f(a_i) = b_i$ for all $i \in I$.*

Proof. This is a special case of a general theorem of algebra, since the class of distributive lattices is equational. A different proof for the case of distributive lattices is given in [1, Th. 1.9].

THEOREM 1.7. *If $\{L_\alpha \mid \alpha \in P\}$ is a family of distributive lattices, and P is a partially ordered set, then $\sum_{\alpha \in P} L_\alpha$ exists.*

Proof. Let $\{x_{\alpha,i} \mid \alpha \in P, i \in L_\alpha\}$ be a family of distinct variables. Let E_α be the set of all inequalities $x_{\alpha,i_1} \cdot \cdots \cdot x_{\alpha,i_m} \leqq x_{\alpha,j_1} + \cdots + x_{\alpha,j_n}$ such that $i_1 \cdots i_m \leqq j_1 + \cdots + j_n$ in L_α. Let E be the union of the $E_\alpha, \alpha \in P$, together with all inequalities $x_{\alpha,i} \leqq x_{\beta,j}$ such that $\alpha < \beta$, $i \in L_\alpha$ and $j \in L_\beta$. By Theorem 1.6, there exists a distributive lattice L generated by a family $\{a_{\alpha,i} \mid \alpha \in P, i \in L_\alpha\}$ satisfying (1.4) and (1.5). We will show $L = \sum_{\alpha \in P} L_\alpha$.

Define $\varphi_\alpha : L_\alpha \rightarrow L$ by $\varphi_\alpha(i) = a_{\alpha,i}$. To show φ_α is a homomorphism, let $i \in L_\alpha$, $j \in L_\alpha$, and let $k = i + j$. Since $i \leqq k$, $j \leqq k$, and $k \leqq i + j$, we have $a_{\alpha,i} \leqq a_{\alpha,k}$, $a_{\alpha,j} \leqq a_{\alpha,k}$, and $a_{\alpha,k} \leqq a_{\alpha,i} + a_{\alpha,j}$, since $\{a_{\alpha,i} \mid i \in L_\alpha\}$ satisfies E_α. It follows that $a_{\alpha,k} = a_{\alpha,i} + a_{\alpha,j}$, and so $\varphi_\alpha(i + j) = \varphi_\alpha(i) + \varphi_\alpha(j)$. Similarly $\varphi_\alpha(ij) = \varphi_\alpha(i)\varphi_\alpha(j)$.

To show that φ_α is injective, suppose $a_{\alpha,i} = a_{\alpha,j}$ but $i \nleqq j$. There exists a prime filter F in L_α such that $i \in F$ and $j \notin F$. Let $M = \{0, 1\}$ be the chain such that $0 < 1$. Let $b_{\alpha,k} = 1$ if $k \in F$, $b_{\alpha,k} = 0$ if $k \in L_\alpha - F$, $b_{\beta,k} = 1$ if $\beta \nleqq \alpha$, $k \in L_\beta$, and $b_{\beta,k} = 0$ if $\beta < \alpha$, $k \in L_\beta$. Since F is a prime filter, $\{b_{\alpha,k} \mid k \in L_\alpha\}$ satisfies all inequalities in E_α. Clearly if $\beta \neq \alpha$, $\{b_{\beta,k} \mid k \in L_\beta\}$ satisfies all inequalities in E_β. If $\beta < \gamma$, then $b_{\beta,k} \leqq b_{\gamma,r}$ for all $k \in L_\beta$, $r \in L_\gamma$. Indeed if not, $b_{\beta,k} = 1$ and $b_{\gamma,r} = 0$, which implies $\gamma \leqq \alpha$ and therefore $\beta < \alpha$, contradicting $b_{\beta,k} = 1$. Thus $\{b_{\beta,k} \mid \beta \in P, k \in L_\beta\}$ satisfies all inequalities in E. By Theorem 1.6, there exists a homomorphism $f : L \rightarrow M$ such that $f(a_{\beta,k}) = b_{\beta,k}$. In particular $f(a_{\alpha,i}) = 1$ and $f(a_{\alpha,j}) = 0$. This contradicts $a_{\alpha,i} = a_{\alpha,j}$.

We now verify (1.1), (1.2) and (1.3). L is generated by $\bigcup_{\alpha \in P} \varphi_\alpha(L_\alpha)$, since L is generated by $\{a_{\alpha,i} \mid \alpha \in P, i \in L_\alpha\}$. Next suppose $\alpha < \beta$, $i \in L_\alpha$, and $j \in L_\beta$. Then $a_{\alpha,i} \leqq a_{\beta,j}$ since $x_{\alpha,i} \leqq x_{\beta,j}$ is in E. To show $a_{\beta,j} \nleqq a_{\alpha,i}$, let $M = \{0, 1\}$ and let $b_{\gamma,k} = 1$ if $\gamma \nleqq \alpha$, $k \in L_\gamma$, and $b_{\gamma,k} = 0$ if $\gamma \leqq \alpha$, $k \in L_\gamma$. Clearly $\{b_{\gamma,k} \mid \gamma \in P, k \in L_\gamma\}$ satisfies all inequalities in E. By Theorem 1.6, there exists a homomorphism $f : L \rightarrow M$ such that $f(a_{\beta,j}) = 1$ and $f(a_{\alpha,i}) = 0$. Therefore $a_{\beta,j} \nleqq a_{\alpha,i}$.

Finally suppose we have homomorphisms f_α as in (1.3). Let $b_{\alpha,i} = f_\alpha(i)$. Obviously $\{b_{\alpha,i} \mid \alpha \in P, i \in L_\alpha\}$ satisfies all inequalities in E. By Theorem 1.6, there exists a homomorphism $f : L \rightarrow M$ such that $f(a_{\alpha,i}) = b_{\alpha,i}$ and therefore $f\varphi_\alpha = f_\alpha$.

In [2, § 9] a construction is given for the free product of a family of distributive lattices which implies that the Stone space of the free product of $\{L_\alpha \mid \alpha \in I\}$ is the Cartesian product of the Stone spaces of the L_α. This statement and the construction are not accurate. The following theorem describes the set of all prime filters of $\sum_{\alpha \in P} L_\alpha$ in the general case where P is partially ordered.

THEOREM 1.8. *Let $\mathfrak{F}$ be the set of all functions f on P such that*

(1.6) *for each $\alpha \in P, f(\alpha)$ is either a prime filter in L_α or*

$f(\alpha) = \varnothing$, *or* $f(\alpha) = L_\alpha$

(1.7)   *if* $\alpha < \beta$ *and* $f(\alpha) \neq \varnothing$, *then* $f(\beta) = L_\beta$

(1.8)   *for some* $\alpha, f(\alpha) \neq \varnothing$, *and for some* $\alpha, f(\alpha) \neq L_\alpha$

There is a one-to-one correspondence between $\mathfrak{F}$ *and the set of all prime filters in* $L = \sum_{\alpha \in P} L_\alpha$ *such that if* F *is the prime filter corresponding to* f, *then* $f(\alpha) = \varphi_\alpha^{-1}(F \cap \varphi_\alpha(L_\alpha))$. *Here* φ_α *is the monomorphism described in Definition 1.1.*

Proof. Let F be a prime filter in $\sum_{\alpha \in P} L_\alpha$, and let f be the function on P such that $f(\alpha) = \varphi_\alpha^{-1}(F \cap \varphi_\alpha(L_\alpha))$. We show $f \in \mathfrak{F}$. Since $\varphi_\alpha(L_\alpha)$ is a sublattice of $\sum_{\alpha \in P} L_\alpha$, (1.6) holds. Conditions (1.7) and (1.8) follow immediately from (1.2) and (1.1) respectively. Conversely, if $f \in \mathfrak{F}$, let $M = \{0, 1\}$, and let $f_\alpha: L_\alpha \to M$ be such that $f_\alpha(x) = 1$ if $x \in f(\alpha)$ and $f_\alpha(x) = 0$ if $x \in L_\alpha - f(\alpha)$. Clearly $\{f_\alpha \mid \alpha \in P\}$ satisfies the hypothesis of (1.3). Therefore there exists a homomorphism $g: L \to M$ such that $g\varphi_\alpha = f_\alpha$. If we set $F = \{x \in L \mid g(x) = 1\}$, F is a prime filter and $f(\alpha) = \varphi_\alpha^{-1}(F \cap \varphi_\alpha(L_\alpha))$.

The following lemma describes the inequalities which hold within an order sum.

LEMMA 1.9.   *Suppose* $L = \sum_{\alpha \in P} L_\alpha$, *and* φ_α *is the monomorphism of Definition 1.1.   Assume* $\alpha_1, \cdots, \alpha_m$ *are distinct members of* $P, \beta_1, \cdots, \beta_n$ *are distinct members of* $P, i_r \in L_{\alpha_r}$ *for* $1 \leq r \leq m$, *and* $j_s \in L_{\beta_s}$ *for* $1 \leq s \leq n$.   *Then*

(1.9) $\varphi_{\alpha_1}(i_1) \cdot \cdots \cdot \varphi_{\alpha_m}(i_m) \leq \varphi_{\beta_1}(j_1) + \cdots + \varphi_{\beta_n}(j_n)$ *if and only if for some* r *and* s *either* $\alpha_r < \beta_s$ *or* $\alpha_r = \beta_s$ *and* $i_r \leq j_s$.

Proof. The condition is obviously sufficient. By changing indices, we may assume $\alpha_1 = \beta_1, \cdots, \alpha_p = \beta_p$, and $\{\alpha_{p+1}, \cdots, \alpha_m\}$ is disjoint from $\{\beta_{p+1}, \cdots, \beta_n\}$, where p may be 0. Let M be the chain $\{0, 1\}$. Suppose the condition fails to hold. Then there exist prime filters F_k in $L_{\alpha_k}, 1 \leq k \leq p$, such that $i_k \in F_k$ and $j_k \notin F_k$. Let f be the function on P such that

$$f(\alpha_k) = F_k, \quad 1 \leq k \leq p,$$
$$f(\alpha_k) = L_{\alpha_k}, \quad p + 1 \leq k \leq m$$
$$f(\alpha) = L_\alpha, \quad \text{if } \alpha > \alpha_k \text{ for some } k, 1 \leq k \leq m$$

and

$$f(\alpha) = \varnothing \quad \text{otherwise.}$$

By Theorem 1.8, there exists a prime filter F in L such that $\varphi_\alpha^{-1}(F \cap \varphi_\alpha(L_\alpha)) = f(\alpha)$. This implies $\varphi_{\alpha_k}(i_k) \in F$ for $1 \leq k \leq m$, and

$\varphi_{\beta_k}(j_k) \notin F$ for $1 \leq k \leq n$. Since F is a prime filter, (1.9) cannot hold.

DEFINITION 1.10. If P is a partially ordered set, we denote by P^* the order sum $\sum_{\alpha \in P} M_\alpha$, where for each α, M_α is a one-element lattice. By Lemma 1.9, the set $\bigcup_{\alpha \in P} \varphi_\alpha(M_\alpha)$ is isomorphic with P. Therefore we will identify the lone member of $\varphi_\alpha(M_\alpha)$ with α, and P will be a generating subset of P^*. If P is a lattice, P will not usually be a sublattice of P^*. P^* is characterized by the property that every order preserving function on P to a distributive lattice M can be extended uniquely to a homomorphism on P^* to M.

THEOREM 1.11. *If Q is any nonempty subset of P, then Q^* is isomorphic with the sublattice N of P^* which is generated by Q.*

Proof. Let f be an order preserving function on Q to a distributive lattice M. By [1, Lemma 1.7.] f can be extended to a homomorphism on N to M if for any finite nonempty subsets S and T of Q, $\pi(S) \leq \Sigma(T)$ implies $\pi(f(S)) \leq \Sigma(f(T))$. This follows immediately from the fact that by Lemma 1.9, $\pi(S) \leq \Sigma(T)$ only when $\alpha \leq \beta$ for some $\alpha \in S, \beta \in T$.

THEOREM 1.12. *Suppose $P = \bigcup_{k \in K} P_k$, where for each k, every element of P_k is incomparable with every element of $P - P_k$. Then P^* is the free product of the P_k^*.*

Proof. By Theorem 1.11, we may assume P_k^* is a sublattice of P^*. Let $\varphi_k: P_k^* \to P^*$ be the inclusion map, and let $f_k: P_k^* \to M$ be a homomorphism into a distributive lattice M. Let $g: P \to M$ be the union of the restrictions $f_k \mid P_k$. Then g is order preserving, and can be extended to a homomorphism $h: P^* \to M$. We have $h \mid P_h^* = f_k$, and the proof is complete.

2. Conditionally implicative lattices.

DEFINITION 2.1. If x and y are members of a lattice, $x \to y$ denotes the largest element z such that $xz \leq y$.

DEFINITION 2.2. A lattice L is *implicative* if $x \to y$ exists for all x, y in L.

It is known that every implicative lattice is distributive and has a largest element. Note also that $x \to y \geq y$.

DEFINITION 2.3. A lattice L is *conditionally implicative* if whenever x, y are in L and $x \nleq y$, then $x \to y$ exists.

A conditionally implicative lattice with a largest element is implicative. If L is a lattice and L' is obtained from L by adding a largest element, then L is conditionally indicative if and only if L' is implicative. It follows that a conditionally implicative lattice is distributive.

LEMMA 2.4. *Let* $x, y,$ *and* z *be members of a distributive lattice. If* $x \rightarrow y$ *and* $x \rightarrow z$ *exist, then* $x \rightarrow yz$ *exists and equals* $(x \rightarrow y)(x \rightarrow z)$. *If* $x \rightarrow z$ *and* $y \rightarrow z$ *exist, then* $(x + y) \rightarrow z$ *exists and equals* $(x \rightarrow z)(y \rightarrow z)$.

Proof. The proof is immediate.

THEOREM 2.5. *If* L_α *is conditionally implicative for each* $\alpha \in P$, *then* $\sum_{\alpha \in P} L_\alpha$ *is conditionally implicative.*

Proof. By (1.1) and Lemma 2.4, we need only show that if $\alpha_1, \cdots, \alpha_m$ are distinct members of $P, \beta_1, \cdots, \beta_n$ are distinct members of P, $i_r \in L_{\alpha_r}$ for $1 \leq r \leq m$, $j_s \in L_{\beta_s}$ for $1 \leq s \leq n$, $x = \varphi_{\alpha_1}(i_1) \cdot \cdots \cdot \varphi_{\alpha_m}(i_m)$, $y = \varphi_{\beta_1}(j_1) + \cdots + \varphi_{\beta_n}(j_n)$, and $x \not\leq y$, then $x \rightarrow y$ exists. Indeed every member of L is a sum of elements of type x as well as a product of elements of type y. We may suppose $\alpha_k = \beta_k$ for $1 \leq k \leq p$ and $\{\alpha_{p+1}, \cdots, \alpha_m\}$ is disjoint from $\{\beta_{p+1}, \cdots, \beta_n\}$. Since $x \not\leq y$, $i_k \not\leq j_k$ for all k. Let

$$z = \sum_{k=1}^{p} \varphi_{\alpha_k}(i_k \rightarrow j_k) + \sum_{k=p+1}^{n} \varphi_{\beta_k}(j_k) \,.$$

Clearly $xz \leq y$ and $y \leq z$. Suppose $w = \varphi_{\gamma_1}(t_1) \cdot \cdots \cdot \varphi_{\gamma_q}(t_q)$ and $xw \leq y$. By Lemma 1.9, either $w \leq y$, hence $w \leq z$, or for some $r, s, \gamma_r = \alpha_s = \beta_s$ and $i_s t_r \leq j_s$. In the second case, $w \leq \varphi_{\gamma_r}(t_r) \leq \varphi_{\alpha_s}(i_s \rightarrow j_s) \leq z$. Since every element of $\Sigma_{\alpha \in P} L_\alpha$ is a sum of elements of type w, it follows that $x \rightarrow y = z$.

REMARK. It is easily seen using Lemma 1.9 that the converse of Theorem 2.5 also holds. Thus Theorem 2.10 below follows directly from Theorem 2.5 and its converse.

COROLLARY 2.6. *A free distributive lattice is conditionally implicative.*

Proof. A free distributive lattice is the free product of one-element lattices.

DEFINITION 2.7. A lattice M is a *retract* of a lattice L if there exist homomorphisms $f: L \rightarrow M$ and $g: M \rightarrow L$ such that $fg = I_M$.

THEOREM 2.8. *If for some* $\beta \in P, L_\beta$ *has a largest and smallest element, then* L_β *is a retract of* $L = \sum_{\alpha \in P} L_\alpha$. *If* P *is trivially ordered, the same holds with no hypothesis on* L_β.

Proof. Let 0 and 1 be the smallest and largest elements of L_β. For each α, define $f_\alpha: L_\alpha \to L_\beta$ by : for each $x \in L_\alpha$,

$$f_\alpha(x) = 1 \quad \text{if} \quad \alpha > \beta$$
$$f_\alpha(x) = x \quad \text{if} \quad \alpha = \beta$$
$$f_\alpha(x) = 0 \quad \text{if} \quad \alpha \not\geq \beta .$$

The family $\{f_\alpha \mid \alpha \in P\}$ satisfies the hypothesis of (1.3). Therefore there exists a homomorphism $f: L \to L_\beta$ such that $f\varphi_\beta = f_\beta = I_{L_\beta}$. In case P is trivially ordered, fix an element x_0 of L_β and define $f_\alpha: L_\alpha \to L_\beta$ by

$$f_\alpha(x) = x_0 \quad \text{if} \quad \alpha \neq \beta$$
$$\qquad\ \ = x \quad \text{if} \quad \alpha = \beta .$$

The rest of the proof is as before.

THEOREM 2.9. *A retract M of a conditionally implicative lattice L is conditionally implicative.*

Proof. Let f and g be homomorphisms as in Definition 2.7. Let x, y be elements of M with $x \not\leq y$. Then $g(x) \not\leq g(y)$, for otherwise $x = fg(x) \leq fg(y) = y$. Let $z = f(g(x) \to g(y))$. We have $g(x)(g(x) \to g(y)) \leq g(y)$. Applying f to both sides, we obtain $xz \leq y$. Now suppose $xw \leq y$. Then $g(x)g(w) \leq g(y)$, and hence $g(w) \leq g(x) \to g(y)$. Applying f, we find $w \leq z$. Thus $z = x \to y$.

THEOREM 2.10. *Let L be the free product of a family $\{L_\alpha \mid \alpha \in I\}$ of distributive lattices. Then L is conditionally implicative if and only if L_α is conditionally implicative for all α.*

Proof. This follows from Theorems 2.5, 2.8 and 2.9.

3. **Order sums and projective lattices.** In contrast to the situation for Boolean algebras, the direct product of two projective distributive lattices is not always projective, [1, Example 8.3]. As a replacement for this theorem, the results of this section show that projectivity is often preserved under order sums.

DEFINITION 3.1. A distributive lattice L is *projective* if for every pair of distributive lattices L_1 and L_2, every homomorphism $h: L \to L_2$, and every epimorphism $f: L_1 \to L_2$, there exists a homomorphism $g: L \to L_1$ such that $fg = h$.

The following lemmas are known and easily proved.

LEMMA 3.2. *A retract of a projective distributive lattice is projective.*

LEMMA 3.3. *If L is projective and $f: L_1 \to L$ is an epimorphism, then there exists a homomorphism $g: L \to L_1$ such that $fg = I_L$. In particular, every projective distributive lattice is a retract of a free distributive lattice.*

THEOREM 3.4. *The free product L of a family $\{L_i \mid i \in I\}$ of distributive lattices is projective if and only if L_i is projective for each $i \in I$.*

Proof. The condition is necessary by Theorem 2.8 and Lemma 3.2. If the condition holds, let M and N be distributive lattices, $f: M \to N$ an epimorphism and $h: L \to N$ a homomorphism. For each $i \in I$, let φ_i be the monomorphism of Definition 1.1. Since L_i is projective, there exists a homomorphism $g_i: L_i \to M$ such that $fg_i = h\varphi_i$. There exists a homomorphism $g: L \to M$ such that $g\varphi_i = g_i$. Now for each i, $fg\varphi_i = fg_i = h\varphi_i$. By (1.1), it follows that $fg = h$.

Theorem 3.4 is a generalization of the following known statement.

LEMMA 3.5. *A free distributive lattice is projective.*

Proof. Since a one-element lattice is projective, the result follows from Theorem 3.4.

THEOREM 3.6. *A projective distributive lattice is conditionally implicative.*

Proof. This follows from Corollary 2.6, Lemma 3.3 and Theorem 2.9.

THEOREM 3.7. *Suppose for each $\alpha \in P$, M_α is a retract of L_α. Then $M = \sum_{\alpha \in P} M_\alpha$ is a retract of $L = \sum_{\alpha \in P} L_\alpha$.*

Proof. By hypothesis there exist homomorphisms $f_\alpha: L_\alpha \to M_\alpha$ and $g_\alpha: M_\alpha \to L_\alpha$ such that $f_\alpha g_\alpha = I_{M_\alpha}$. Let $\varphi_\alpha: L_\alpha \to L$ and $\psi_\alpha: M_\alpha \to M$ be the monomorphisms of Definition 1. Let $h_\alpha = \psi_\alpha f_\alpha$. By (1.3) there exists a homomorphism $f: L \to M$ such that $f\varphi_\alpha = h_\alpha$, $\alpha \in P$. Similarly there exists a homomorphism $g: M \to L$ such that $g\psi_\alpha = \varphi_\alpha g_\alpha$. For each α, we have $fg\psi_\alpha = f\varphi_\alpha g_\alpha = \psi_\alpha f_\alpha g_\alpha = \psi_\alpha$. Since M is generated by $\bigcup_{\alpha \in P} \psi_\alpha(M_\alpha)$, it follows that $fg = I_M$.

LEMMA 3.8. *If P is a partially ordered set, then P^* is projective if and only if for each $\alpha \in P$ there exists a finite sequence*

$S_{\alpha,0}, \cdots, S_{\alpha,p(\alpha)}$ *of nonempty finite subsets of* P *such that*

(3.1) $\alpha \in S_{\alpha,p(\alpha)}$, *and every member of* $S_{\alpha,p(\alpha)}$ *is* $\geq \alpha$.

(3.2)   *for each* $r, S_{\alpha,r}$ *has a member* $\leq \alpha$.

(3.3)   *if* $\alpha < \beta$, *then for every* r *there is an* s *such that* $S_{\alpha,r} \supseteqq S_{\beta,s}$.

Proof. By [1, Th. 5.1], P^* is projective if and only if for each α there exist nonempty finite subsets $S_{\alpha,0}, \cdots, S_{\alpha,p(\alpha)}$ of P such that (3.3) holds and

(3.4) $$\alpha = \sum_{r=0}^{p(\alpha)} \pi(S_{\alpha,r}),$$

where the products and sum are relative to P^*. By Lemma 1.9, (3.4) is equivalent to (3.2) and

(3.5)   for some r, every element of $S_{\alpha,r}$ is $\geq \alpha$.

By renumbering the indices, we obtain (3.1) from (3.2) and (3.5).

THEOREM 3.9.   *Let* $\{L_\alpha \mid \alpha \in P\}$ *be a family of distributive lattices each of which have a smallest and largest element. Then* $L = \sum_{\alpha \in P} L_\alpha$ *is projective if and only if* L_α *is projective for each* α, *and* P^* *is projective.*

Proof. Suppose L is projective. By Lemma 3.2 and Theorem 2.8, L_α is projective for each α. Since a one-element lattice is a retract of any lattice, the projectivity of P^* follows from Lemma 3.2 and Theorem 3.7.

For the converse, let N be a free distributive lattice which is freely generated by a family $\{d_{\alpha,i} \mid \alpha \in P, i \in L_\alpha\}$ of distinct elements. Let N_α be the sublattice generated by $\{d_{\alpha,i} \mid i \in L_\alpha\}$, let $f_\alpha: N_\alpha \to \varphi_\alpha(L_\alpha)$ be the homomorphism such that $f_\alpha(d_{\alpha,i}) = \varphi_\alpha(i)$, $i \in L_\alpha$, and let $f: N \to L$ be the extension of the f_α. By Corollary 3.5 and Lemma 3.3, there exist homomorphisms $g_\alpha: \varphi_\alpha(L_\alpha) \to N_\alpha$ such that $f_\alpha g_\alpha = I_{\varphi(L_\alpha)}$. By Lemma 3.2, we need only find a homomorphism $g: L \to N$ such that $fg = I_L$.

For each α, there exist sets $S_{\alpha,1}, \cdots, S_{\alpha,p(\alpha)}$ as in Lemma 3.8. For $i \in L_\alpha$, let

$$b_{\alpha,i,\beta} = \varphi_\alpha(i) \quad \text{if} \quad \beta = \alpha$$
$$= \varphi_\beta(0_\beta) \quad \text{if} \quad \beta > \alpha$$
$$= \varphi_\beta(1_\beta) \quad \text{if} \quad \beta \ngeqq \alpha,$$

where $0_\beta, 1_\beta$ are the smallest and largest elements of L_β. Let

$$c_{\alpha,i} = \sum_{r=1}^{p(\alpha)} \prod_{\beta \in S_{\alpha,r}} g_{\beta}(b_{\alpha,i,\beta}) \ .$$

We wish to show that $\{c_{\alpha,i} \mid \alpha \in P, i \in L_{\alpha}\}$ satisfies all the inequalities of the set E defined in the proof of Theorem 1.7. First, suppose $i_1 \cdot \ \cdots \ \cdot i_m \leqq j_1 + \cdots + j_n$, where the elements involved are in L_{α}. For each $i \in L_{\alpha}$, we have $c_{\alpha,i} = g_{\alpha}(\varphi_{\alpha}(i))P_{\alpha} + Q_{\alpha}$, where P_{α} and Q_{α} are independent of i. Therefore

$$\begin{aligned}
c_{\alpha,i_1} \cdot \ \cdots \ \cdot c_{\alpha,i_m} &= g_{\alpha}\varphi_{\alpha}(i_1) \cdot \ \cdots \ \cdot g_{\alpha}\varphi_{\alpha}(i_m)P_{\alpha} + Q_{\alpha} \\
&\leqq [g_{\alpha}\varphi_{\alpha}(j_1) + \cdots + g_{\alpha}\varphi_{\alpha}(j_n)]P_{\alpha} + Q_{\alpha} \\
&= c_{\alpha,j_1} + \cdots + c_{\alpha,j_n} \ .
\end{aligned}$$

Next, suppose $\alpha < \beta$, $i \in L_{\alpha}$, and $j \in L_{\beta}$. To show $c_{\alpha,i} \leqq c_{\beta,j}$, first observe that for any $\gamma \in P$, $b_{\alpha,i,\gamma} \leqq b_{\beta,j,\gamma}$ (consider the cases $\gamma > \beta$, $\gamma = \beta$, and $\gamma \not\geqq \beta$). By (3.3) for each r, $0 \leqq r \leqq p(\alpha)$, $S_{\alpha,r} \supseteqq S_{\beta,s}$ for some s. Therefore for every r,

$$\prod_{\gamma \in S_{\alpha,r}} g_{\gamma}(b_{\alpha,i,\gamma}) \leqq \prod_{\gamma \in S_{\beta,s}} g_{\gamma}(b_{\alpha,i,\gamma}) \leqq \prod_{\gamma \in S_{\beta,s}} g_{\gamma}(b_{\beta,j,\gamma}) \leqq c_{\beta,j} \ .$$

Hence $c_{\alpha,i} \leqq c_{\beta,j}$.

By Theorem 1.6, there exists a homomorphism $g : L \to N$ such that $g(\varphi_{\alpha}(i)) = c_{\alpha,i}$ for all $\alpha \in P, i \in L_{\alpha}$. Now, since f extends each f_{β}, we have for $\alpha \in P, i \in L_{\alpha}$,

$$(3.6) \qquad fg(\varphi_{\alpha}(i)) = f(c_{\alpha,i}) = \sum_{r=1}^{p(\alpha)} \prod_{\beta \in S_{\alpha,r}} b_{\alpha,i,\beta} \ .$$

By (3.1) and (3.2), it is easily seen that the right side of (3.6) reduces to $\varphi_{\alpha}(i)$. Using (1.1), we conclude that $fg = I_M$.

We have been unable to find workable necessary and sufficient conditions for the projectivity of P^*. It is necessary that P contain no uncountable chain, since no projective lattice can contain such a chain [1, Th. 4.2]. On the other hand, the countability of P is a sufficient condition, as shown by the following.

THEOREM 3.10. *If P is countable, and L_{α} is a countable pro-jective distributive lattice for each $\alpha \in P$, then $L = \sum_{\alpha \in P} L_{\alpha}$ is projective.*

Proof. Let $N, N_{\alpha}, f_{\alpha}, f$, and g_{α} be as in the proof of Theorem 3.9. Arrange the members of $\bigcup_{\alpha \in P} \varphi_{\alpha}(L_{\alpha})$ in a sequence $a_1, a_2, \cdots$ without repetitions. For each n, let α_n be the α such that $a_n \in \varphi_{\alpha}(L_{\alpha})$. We define inductively:

$$c_1 = g_{\alpha_1}(a_1) \ ,$$

and for $n > 1$,

$$c_n = g_{\alpha_n}(a_n) \prod \{c_r \mid r < n, \alpha_r > a_n\} + \sum \{c_r \mid r < n, \alpha_r < \alpha_n\} \,.$$

We wish to show there exists a homomorphism $g: L \to N$ such that $g(a_n) = c_n$ for all n. By Theorem 1.6, this will be the case if we show:

(3.7)   If $\alpha_m < \alpha_n$, then $c_m \leqq c_n$, and

(3.8)
$$\text{If } \alpha_{m_1} = \cdots = \alpha_{m_p} = \alpha_{n_1} = \cdots = \alpha_{n_q} \,, \text{ and}$$
$$a_{m_1} \cdot \, \cdots \cdot a_{m_p} \leqq a_{n_1} + \cdots + a_{n_q} \,, \text{ then}$$
$$c_{m_1} \cdot \, \cdots \cdot c_{m_p} \leqq c_{n_1} + \cdots + c_{n_q} \,.$$

We prove (3.7) by proving the following by induction on n:

(3.9)
     If $m < n$, then $\alpha_m < \alpha_n$ implies $c_m \leqq c_n$, and
$\alpha_m > \alpha_n$ implies $c_m \geqq c_n$.

For $n = 1$, this holds vacuously. Suppose (3.9) holds for all $n < p$, and $p > 1$. If $m < p$ and $\alpha_m < \alpha_p$, then $c_m \leqq \sum \{c_r \mid r < p, \alpha_r > \alpha_p\} \leqq c_p$, where the second inequality holds by definition of c_p. Suppose $m < p$ and $\alpha_m > \alpha_p$. Then c_m is one of the factors of $g_{\alpha_p}(a_p) \prod \{c_r \mid r < p, \alpha_r > \alpha_p\}$. Hence to show $c_m \geqq c_p$, we need only show $c_m \geqq c_r$ whenever $r < p$ and $\alpha_r < \alpha_p$. Since $\alpha_m > \alpha_r$ and $m < p, r < p$, this follows from the induction hypothesis.

As for (3.8), let α be the common value of $\alpha_{m_1}, \cdots, \alpha_{n_q}$. Replace each of $c_{m_1}, \cdots, c_{m_p}$ by its defining formula and distribute the product of these as a sum of products. We obtain $c_{m_1} \cdot \, \cdots \cdot c_{m_p} = A + B$, where

$$A = g_\alpha(a_{m_1}) \prod \{c_r \mid r < i_1, \alpha_r > \alpha\} \cdot \, \cdots \cdot g_\alpha(a_{m_p}) \prod \{c_r \mid r < i_p \mid \alpha_r > \alpha\}$$

and B is a sum of products each of which has a factor c_r with $\alpha_r < \alpha$. By (3.7), $B \leqq c_{n_1}$. It remains to show $A \leqq c_{n_1} + \cdots + c_{n_q}$. Using (3.7), we have

$$c_{m_1} \cdot \, \cdots \cdot c_{m_p} \leqq \prod \{c_r \mid r < j_k, \alpha_k > \alpha\} \quad \text{for} \quad 1 \leqq k \leqq q \,.$$

Therefore

$$c_{n_1} + \cdots + c_{n_q} \geqq \sum_{k=1}^{q} [g_\alpha(a_{n_k}) \prod \{c_r \mid r > j_k, \alpha_k < \alpha\}]$$

$$\geqq c_{m_1} \cdot \, \cdots \cdot c_{m_p} \sum_{k=1}^{q} g_\alpha(a_{n_k})$$

$$\geqq c_{m_1} \cdot \, \cdots \cdot c_{m_p} \prod_{k=1}^{p} g_\alpha(a_{m_k})$$

$$\geqq c_{m_1} \cdot \, \cdots \cdot c_{m_p} \cdot A = A \,.$$

This proves the existence of the homomorphism g. By Lemma 3.2, we have only to prove $fg = I_L$. This will hold if we show

$$(3.10) \qquad\qquad fg(a_n) = a_n , \qquad \text{for all } n.$$

We have $fg(a_1) = f(c_1) = fg_{\alpha_1}(a_1) = f_{\alpha_1}g_{\alpha_1}(a_1) = a_1$. Suppose (3.10) holds for $n < p$, and $p > 1$. Then

$$fg(a_p) = fg_{\alpha_p}(a_p) \prod \{fg(a_r) \mid r < p, \alpha_r > \alpha_p\}$$
$$+ \sum \{fg(a_r) \mid r < p, \alpha_r < \alpha_p\} ,$$

which by the induction hypothesis,

$$= a_p \prod \{a_r \mid r < p, \alpha_r > \alpha_p\} + \sum \{a_r \mid r < p, \alpha_r < \alpha_p\}$$
$$= a_p ,$$

by (1.2).

COROLLARY 3.11. *If P is countable, then P^* is projective.*

The next theorem gives another sufficient condition for the projectivity of P^*.

THEOREM 3.12. *Let P be partially ordered set such that:*

$$(3.11) \qquad \textit{for each } \alpha, \{\beta \mid \beta \geqq \alpha\} \textit{ is well ordered by } \leqq, \textit{ and}$$

$$(3.12) \qquad \textit{if } \alpha < \beta, \textit{ then } \{\gamma \mid \alpha \leqq \gamma \leqq \beta\} \textit{ is finite.}$$

Proof. For each α, let $f(\alpha) = \alpha$ if α is a maximal member of P, and $f(\alpha) = $ the immediate successor of α otherwise. Let $f^0 = I_P$, and for an integer $p > 1$, $f^P = ff^{p-1}$. Let us write $\alpha \sim \beta$ whenever for some $\gamma, \gamma \geqq \alpha$ and $\gamma \geqq \beta$. By (3.11), $\sim$ is an equivalence relation. Let $\{P_k \mid k \in K\}$ be the set of all $\sim$ equivalence classes. By Theorem 1.12, P^* is the free product of $\{P_k^* \mid k \in K\}$. Therefore by Theorem 3.4, we may assume $\alpha \sim \beta$ for all α, β in P.

Fix an element α_0 of P. For each α, let $p(\alpha)$ be the smallest integer $n \geqq 0$ such that $f^n(\alpha_0) \geqq \alpha$, and let $q(\alpha)$ be the smallest integer $n \geqq 0$ such that $f^n(\alpha) = f^{p(\alpha)}(\alpha_0)$. For $0 \leqq r \leqq p(\alpha)$, let

$$S_{\alpha,r} = \{f^r(\alpha_0)\} \cup \{f^n(\alpha_0) \mid 0 \leqq n \leqq q(\alpha) - 1\} .$$

Conditions (3.1) and (3.2) of Lemma 3.8 clearly hold. To verify (3.3), we need only show that for each r there is an s such that $S_{\alpha,r} \supseteqq S_{f(\alpha),s}$. If $\alpha_0 \not\leqq \alpha$, then $p(f(\alpha)) = p(\alpha)$ and $q(f(\alpha)) = q(\alpha) - 1$, and for each $r, S_{\alpha,r} \supseteqq S_{f(\alpha),s}$. If $\alpha_0 \leqq \alpha$, then $\alpha = f^{p(\alpha)}(\alpha_0)$, $p(f(\alpha)) =$

$p(\alpha) + 1$, $q(f(\alpha)) = q(\alpha) = 0$, and $S_{\alpha,r} = S_{f(\alpha),r}$ for each $r \leq p(\alpha)$. Thus by Lemma 3.8, P^* is projective.

We return to the question of the projectivity of order sums of projective lattices.

THEOREM 3.13. *If L_0 and L_1 are free distributive lattices, L_0 is uncountable, and L_1 is infinite, then the ordinal sum $L_0 \oplus L$, is not projective.*

Proof. We may assume L_0 and L_1 are disjoint. Let L_0 and L_1 be freely generated by $A = \{a_\alpha \mid \alpha \in I\}$ and $B = \{b_\beta \mid \beta \in J\}$ respectively. If $L_0 \oplus L_1$ is projective, then by [1, Corollary 5.2, and remark following Th. 5.1] for each α and β there exist finite sequences $S_{\alpha,0}, \cdots, S_{\alpha,p(\alpha)}$ and $S_{\beta,0}, \cdots, S_{\beta,p}(\beta)$ of non-empty finite subsets of $A \cup B$ such that:

(3.13)
$$a_\alpha = \sum_{r=0}^{p(\alpha)} \prod (S_{\alpha,r})$$

(3.14)
$$b_\beta = \sum_{r=0}^{p(\beta)} \prod (S_{\beta,r})$$

(3.15)
$$\text{if } r \neq s, S_{\beta,r} \nsubseteq S_{\beta,s}$$

(3.16)   for each α, β and each r, $0 \leq r \leq p(\alpha)$, there exists an s, $0 \leq s \leq p(\beta)$, such that $S_{\alpha,r} \supseteq S_{\beta,s}$.

Since $a_\alpha < b_\beta$ for all α, β and B is independent, it follows from (3.14) that $S_{\beta,r} = \{b_\beta\}$ for some r, say $r = 0$. If $0 < r \leq p(\beta)$, $\prod (S_{\beta,r}) \leq b_\beta$ and $b_\beta \notin S_{\beta,r}$ by (3.15). Therefore by the independence of B, $S_{\beta,r} \cap A \neq \varnothing$ for $r > 0$.

Let $\beta_1, \beta_2, \cdots$ be a sequence of distinct members of J. For each integer $n > 0$, let

$$\Gamma_n = \bigcup_{r=1}^{p(\beta_n)} S_{\beta_n,r} \cap A \; .$$

Since Γ_n is countable, there is an $\alpha \in A$ such that $a_\alpha \notin \Gamma_n$. If we show

(3.17)
$$b_{\beta_n} \in \bigcup_{r=0}^{p(\alpha)} S_{\alpha,r}$$

for all n, we will have a contradiction since this union is finite.

Suppose (3.17) fails for some n. By (3.16) for $0 \leq r \leq p(\alpha)$, either $S_{\alpha,r} \supseteq S_{\beta_n,0} = \{b_{\beta_n}\}$, or $S_{\alpha,r} \supseteq S_{\beta_n,s}$ for some $s > 0$. The first case is impossible by assumption. Hence for each r, $S_{\alpha,r}$ contains an element a_α of $S_{\beta_n,s} \cap A$. By (3.13), $a_\alpha \leq \sum_{r=0}^{p(\alpha)} a_{\alpha_r}$. Since A is inde-

pendent, we have $\alpha = \alpha_r$ for some r and therefore $a_\alpha \in \Gamma_n$, which is a contradiction.

DEFINITION 3.14. If L is a lattice, let $D(L)$ be the dual of L, that is, L with reverse order.

The dual of a projective (free) distributive lattice is projective (free).

In contrast to Theorem 3.13, we have the following.

THEOREM 3.15. *If L_0 and L_1 are projective and either L_0 has a largest element or L_1 has a smallest element, then $L = L_0 \oplus L_1$ is projective.*

Proof. We may assume L_0 and L_1 are disjoint. Let N be a free distributive lattice with distinct free generators $\{d_{0,i} \mid i \in L_0\} \cup \{d_{1,i} \mid i \in L_1\}$. For $r = 0, 1$, let N_r be the sublattice generated by $\{d_{r,i} \mid i \in L_r\}$, and let $f_r: N_r \to L_r$ be the homomorphism such that $f_r(d_{r,i}) = i$. Let $f: N \to L$ be the homomorphism which extends f_0 and f_1. By Lemma 3.3, there exists homomorphisms $g_r = L_r \to N_r$ such that $f_r g_r = I_{L_r}$.

Suppose L_0 has a largest element 1. Define $g: L \to N$ by

$$
\begin{aligned}
g(x) &= g_0(x) \,, &&\text{if } x \in L_0 \\
&= g_0(1) + g_1(x) \,, &&\text{if } x \in L_1 \,.
\end{aligned}
$$

It is easily seen that g is a homomorphism and $fg = I_L$. Hence L is projective. If L_1 has a smallest element, then $D(L_1) \oplus D(L_0)$ is projective, since $D(L_1)$ has a largest element. Therefore $L = D(D(L_1) \oplus D(L_0))$ is projective.

COROLLARY 3.16. *Let L_0 and L_1 be free distributive lattices. Then $L = L_0 \oplus L_1$ is projective if and only if either*
(i) *L_0 and L_1 are countable, or*
(ii) *L_0 or L_1 is finite.*

Proof. If (i) holds, L is projective by Theorem 3.10. If (ii) holds, L is projective by Theorem 3.15. Suppose neither (i) nor (ii) holds. Then one of L_0, L_1 is uncountable and the other is infinite. In this case L is not projective by Theorem 3.13 and duality.

COROLLARY 3.17. *Let L_0 and L_1 be uncountable free distributive lattices, and $\{a\}$ be a one-element lattice. Then $L_0 \oplus L_1$ is not projective, but $L_0 \oplus \{a\} \oplus L_1$ is projective.*

Proof. This follows from Theorems 3.13 and 3.15 and the associativity of ordinal sums.

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