ON THE STONE-WEIERSTRASS APPROXIMATION THEOREM 
FOR VALUED FIELDS

DAVID GEOFFREY CANTOR
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THEOREM FOR VALUED FIELDS

DAVID G. CANTOR

Let $X$ be a compact topological space, $L$ a non-Archimedean
rank 1 valued field and $\mathcal{F}$ a uniformly closed $L$-algebra of
$L$-valued continuous functions on $X$. Kaplansky has shown
that if $\mathcal{F}$ separates the points of $X$, then either $\mathcal{F}$ consists of
all $L$-valued continuous functions on $X$ or else all of them
which vanish on one point in $X$. In this paper analogous
results are obtained, in the case that a group of transforma-
tions acts both on $X$ and $L$, for the invariant $L$-valued con-
tinuous functions on $X$.

If $L$ and $K$ are fields such that $L \subset K$ and $L/K$ is normal, we let
$\text{Aut}(L/K)$ denote the group of automorphisms of $L$ which leave every
element of $K$ fixed, and we give $\text{Aut}(L/K)$ the Krull topology; a basis
for the open neighborhoods of the identity of $\text{Aut}(L/K)$ is given by
subgroups of the form

$$\{\sigma \in \text{Aut}(L/K) : \sigma x = x \text{ if } x \in L_i\}$$

where $L_i$ is a finite extension of $K$ contained in $L$.

Now suppose that $L$ is a non-Archimedean field with a (multiplica-
tive) rank 1 valuation, denoted $| | [1]$. Suppose $K$ is a subfield of $L
such that $L/K$ is both normal and separable. Denote by $L_\omega$ a com-
pletion of $L$ and let $K'$ be the closure of $K$ in $L_\omega$. Put $L' = LK'
(the composite field generated by $L$ and $K'$ in $L_\omega$) and note that $K'$
is dense in $K'$. It is clear that $L'/K'$ is normal and separable. If $\sigma \in
\text{Aut}(L'/K')$, then, since $K'$ is complete, $|\sigma x| = |x|$ for each $x \in L'
so that $\sigma$ is a continuous map of $L'$ onto itself; furthermore the re-
striction of $\sigma$ to $L$, $\sigma|_L \in \text{Aut}(L/K)$. Finally suppose that $X$ is a com-
 pact topological space for which there exists a continuous map $(\sigma, x) \mapsto \sigma x
of $\text{Aut}(L'/K') \times X \to X$ satisfying $\sigma_1(\sigma_2 x) = (\sigma_1 \sigma_2) x$
if $\sigma_1, \sigma_2 \in \text{Aut}(L'/K')$, $x \in X$ and satisfying $ex = x$ if $e$ is the identity of $\text{Aut}(L'/K')$ and
$x \in X$. It is immediate that if $\sigma \in \text{Aut}(L'/K')$ then the map $x \mapsto \sigma x
of $X \to X$ is a homeomorphism of $X$. We shall call a set $Y \subset X in-
variant if $\text{Aut}(L'/K') Y = Y$. Denote by $C_{L/K}(X)$ the set of $L$-valued
continuous functions $f$ on $X$ satisfying $f(\sigma x) = \sigma f(x)$ for all $x \in X$ and
$\sigma \in \text{Aut}(L'/K'); C_{L/K}(X)$ is a $K$-algebra. If $E$ is any valued field,
denote by $C_{E}(X)$ the continuous $E$-valued functions on $X$ and give
$C_{E}(X)$ the sup-norm topology. Clearly $C_{L}(X) \supset C_{L/K}(X) \supset C_{K}(X)$.

**Theorem 1.** Suppose $\mathcal{F}$ is a closed (in the sup-norm) $K$sub-
algebra of $C_{L/K}(X)$ which separates the points of $X$ (i.e., if $x, y \in X$ and $x \neq y$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$). Then either $\mathcal{F} = C_{L/K}(X)$ or there exists $x_0 \in X$ such that

$$\mathcal{F} = \{f \in C_{L/K}(X) : f(x_0) = 0\}.$$ 

In the latter case the set $\{x_0\}$ is invariant.

**Proof.** Let $\mathcal{F}'$ be the uniform closure of the $K'$ algebra of functions generated by $g$ in $C_{L'}(X)$; since $K$ is dense in $K'$, $\mathcal{F}$ is dense in $\mathcal{F}'$ and hence it suffices to prove that $\mathcal{F}' = C_{L'/K'}(X)$ or that $\mathcal{F}' = \{f \in C_{L'/K'}(X) : f(x_0) = 0\}$. Thus we may assume without loss of generality that $K = K'$ and $L = L'$. We assume first that for each $x \in X$, there exists $f \in \mathcal{F}$ such that $f(x) \neq 0$

**Lemma 2.** Assuming the hypotheses of Theorem 1, if $x_0 \in X$ and $g \in C_{L/K}(X)$, there exists $f \in \mathcal{F}$ such that $f(x_0) = g(x_0)$.

**Proof.** Put $L_i = \{h(x_0) : h \in \mathcal{F}\}$; clearly $L_i$ is a $K$-subalgebra of $L$ containing a nonzero element of $L$. Suppose $c \in L_i$ and $c \neq 0$; $c$ satisfies a polynomial equation $\sum_{i=0}^n a_i c^i = 0$, where the $a_i \in K$ and $a_0 \neq 0$. Then $a_0 \in L_i$ and hence $K = Ka_0 \subset L_i$. It follows that $L_i$ is a subfield of $L$.

Put

$$H = \{\sigma \in \text{Aut}(L/K) : \sigma x_0 = x_0\};$$

$H$ is a closed subgroup of $\text{Aut}(L/K)$ which fixes every element of $L_i$ and also fixes $g(x_0)$. Now if $\sigma \in \text{Aut}(L/K) - H$, then $x_0 \neq \sigma x_0$, and there exists $h \in \mathcal{F}$ such that $h(x_0) \neq h(\sigma x_0)$ or $h(x_0) \neq \sigma h(x_0)$. Equivalently, if $\sigma \in \text{Aut}(L/K)$ fixes every element of $L_i$, then $\sigma \in H$. Thus $L_i$ is the fixed field of the closed subgroup $H$. As $H$ fixes $g(x_0)$, we have $g(x_0) \in L_i$, and there exists $f \in \mathcal{F}$ such that $f(x_0) = g(x_0)$.

**Lemma 3.** Assuming the hypotheses of Theorem 1, $X$ is totally disconnected.

**Proof.** Since $\mathcal{F}$ separates points, $X$ is Hausdorff. Now take $x_0 \in X$ and an open neighborhood $U$ of $x_0$. For each $y \in U$, there exists $f_y \in \mathcal{F}$ such that $f_y(x_0) \neq f_y(y)$. Put $\varepsilon_y = |f_y(x_0) - f_y(y)|$, and let

$$U_y = \{x \in X : |f_y(x) - f_y(x_0)| < \varepsilon_y/2\}$$

and

$$V_y = \{x \in X : |f_y(x) - f_y(y)| < \varepsilon_y/2\};$$

$U_y$ and $V_y$ are disjoint open and closed subsets of $X$ with $x_0 \in U_y$. 

The $V_x$ cover the compact set $X - U$ and hence there exists a finite number, say $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$ whose union contains $X - U$. Then $\bigcap U_{x_i}$ is an open and closed neighborhood of $x$ contained in $U$.

**Lemma 4.** Assuming the hypotheses of Theorem 1, suppose $V$ is an open and closed invariant subset of $X$. Then the characteristic function of $V$ is in $\mathcal{G}$.

**Proof.** By the Kaplansky-Stone-Weierstrass Theorem [2] and Lemma 3, the characteristic function of $V$ is in the uniform closure of the $L$-subalgebra of $C_L(X)$ generated by $\mathcal{G}$. Hence, if $\epsilon > 0$, there exists $f \in C_L(X)$ such that $f = \sum a_i h_i$ where the $a_i \in L$ and the $h_i \in \mathcal{G}$ and such that $|f(y) - 1| < \epsilon$ if $y \in V$ while $|f(y)| < \epsilon$ if $y \notin V$. Let $L \subseteq L_1$ be the smallest normal extension field of $K$ containing all of the $a_i$; $L_1$ is a finite algebraic extension of $K$ and hence $\text{Aut}(L_1/K)$ is finite. As $\text{Aut}(L_1/K)$ is a homomorphic image of $\text{Aut}(L/K)$, there exist representatives $\sigma_1, \sigma_2, \ldots, \sigma_n$ of $\text{Aut}(L_1/K)$ in $\text{Aut}(L/K)$ and the set of restrictions $\{\sigma_i|_{L_1}: 1 \leq i \leq n\}$ is $\text{Aut}(L_1/K)$. If $\sigma \in \text{Aut}(L/K)$, put $f^\sigma = \sum (\sigma a_i) h_i$. Then if $y \in X$,

$$f^\sigma(y) = \sum_{i=1}^n (\sigma a_i) h_i(\sigma y) = \sigma(\sum_{i=1}^n a_i h_i(\sigma^{-1}y)) = \sigma f(\sigma^{-1}y).$$

As $\sigma^{-1}V = V$, $|f^\sigma(y) - 1| < \epsilon$ if $y \in V$, while $|f^\sigma(y)| < \epsilon$ if $y \notin V$. Put $g = \prod f^\sigma; \text{then } g \in \mathcal{G}$ and $|g(y) - 1| < \epsilon$ if $y \in V$ while $|g(y)| < \epsilon$ if $y \notin V$. Thus letting $\epsilon \to 0$, we see that the characteristic function of $V$ is in $\mathcal{G}$.

**Proof of Theorem 1 (concluded).** Suppose $f \in C_{L/K}(X)$ and $\epsilon > 0$. For each $x \in X$, there exists by Lemma 2, $g_x \in \mathcal{G}$ such that $g_x(x) = f(x)$. Let $U_x$ be an open and closed neighborhood of $x$ such that $|g_x(y) - f(y)| < \epsilon$ whenever $y \in U_x$. Put $V_x = \text{Aut}(L/K) U_x$; clearly $V_x$ is invariant. As $V_x$ is the union of the open sets $\sigma U_x, \sigma \in \text{Aut}(L/K), V_x$ is open, and since it is the continuous image of the compact set $\text{Aut}(L/K) \times U_x$, it is compact. If $y \in V_x$, there exists $\sigma \in \text{Aut}(L/K)$ such that $\sigma y \in U_x$. Then

$$|g_x(y) - f(y)| = |\sigma(g_x(y) - f(y))| = |g_x(\sigma y) - f(\sigma y)| < \epsilon.$$

The $V_x$ are open sets which cover $X$. Hence a finite number, say $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$ cover $X$. Put $D_i = V_{x_i}$ and for $2 \leq i \leq n$, put
Each $D_i$ is open and closed, and invariant; hence by Lemma 4, the characteristic function $h_i$ of $D_i$ is in $\mathfrak{F}$. In addition the $D_i$ are disjoint and $\bigcup_{i=1}^n D_i = X$. Now put

$$g = \sum_{i=1}^n h_ig_{x_i},$$

so that $g \in \mathfrak{F}$. If $y \in X$, then there exists $j$ such that $y \in D_j \subset V_{x_j}$; then $g(y) = g_{x_j}(y)$. As $|g_{x_j}(y) - f(y)| < \varepsilon$, $|g(y) - f(y)| < \varepsilon$. Letting $\varepsilon \to 0$ shows that $f \in \mathfrak{F}$. Finally, if there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathfrak{F}$, let $\mathfrak{F}$ be the $K$-algebra obtained from $\mathfrak{F}$ by adjoining the $K$-valued constant functions. Then if $g \in C_{L/K}(X)$ satisfies $g(x_0) = 0$, and $\varepsilon > 0$, there exists by what we have proved $f_1 \in \mathfrak{F}$, such that $|f_1(x) - f(x)| < \varepsilon$ for all $x \in X$. Then $f_1 = f + a$, where $f \in \mathfrak{F}$ and $a \in K$. Now $|a| = |f_1(x_0)| < \varepsilon$, hence $|f(x) - g(x)| < \varepsilon$ for all $x \in X$. Letting $\varepsilon \to 0$ shows that $g \in \mathfrak{F}$.

**COROLLARY 5.** Suppose that $C_{L/K}(X)$ separates the points of $X$ and that $I$ is a closed ideal of the $K$-algebra $C_{L/K}(X)$. Then there exists a closed invariant set $Y \subset X$ such that

$$I = \{f \in C_{L/K}(X) : f(Y) = \{0\}\}.$$

**Proof.** Put $Y = \bigcap_{x \in I} \{x : f(x) = 0\}$. Then $Y$ is a closed invariant subset of $X$. If $x_1, x_2 \in X - Y$ and $x_1 \neq x_2$, then there exists $f \in I$ such that $f(x_1) \neq 0$. If $f(x_1) = f(x_2)$, let $g$ be the constant function 1, while if $f(x_1) = f(x_2)$, choose $g \in C_{L/K}(X)$ such that $g(x_1) \neq g(x_2)$. Then in either case the function $h = gf \in I$ and $h(x_1) \neq h(x_2)$. Now let $X_1$ be the topological space obtained from $X$ by identifying the points of $Y$, and let $p$ be the projection from $X$ to $X_1$. Then $p$ is continuous and if $x_1, x_2 \in X$, we have $p(x_1) = p(x_2)$ if and only if either $x_1 = x_2$ or $x_1, x_2 \in Y$. A basis for the open neighborhoods of a point $x \in X_1$ is given by sets of the form $p(V)$, where $V$ is an open neighborhood of $p^{-1}(x)$ in $X$. If $\sigma \in \text{Aut}(L'/K')$ and $x \in X_1$, we define $\sigma x = p(\sigma p^{-1}(x))$; this is well defined and yields a continuous map $(\sigma, x) \mapsto \sigma x$ of $\text{Aut}(L'/K') \times X_1 \to X_1$. Denote by $C_{L/K}(X, Y)$ the $K$-algebra of $f \in C_{L/K}(X)$ which are constant on $Y$. If $f \in C_{L/K}(X, Y)$ define $pf \in C_{L/K}(X_1)$ by $(pf)(x) = f(p^{-1}(x))$; this is well defined and yields a norm preserving isomorphism between $C_{L/K}(X, Y)$ and $C_{L/K}(X_1)$. Put $pI = \{pf : f \in I\}; pI$ is a uniformly closed $K$-subalgebra which separates the points of $X_1$, and every function $pf \in pI$ vanishes on $p(Y)$; hence by Theorem 1, $pI$ consists of all $f \in C_{L/K}(X_1)$ which vanish on $p(Y)$. Thus $I$ consists of all $f \in C_{L/K}(X)$ which vanish on $Y$. \[476\text{ DAVID G. CANTOR}\]
COROLLARY 6. Suppose that \( C_{L|K}(X) \) separates the points of \( X \). Then the maximal ideals of the \( K \)-algebra \( C_{L|K}(X) \) are precisely the sets of the form
\[
\{ f \in C_{L|K}(X) : f(x_0) = 0 \}
\]
where \( x_0 \in X \).

The following theorem permits the extension of Theorem 1 and its corollaries to certain subsets of \( X \).

THEOREM 7. Suppose \( Y \) is a closed subset of \( X \) and \( \text{Aut}(L'/K')Y = X \). Then each continuous \( K \)-valued function \( f \) on \( Y \), satisfying \( f(\sigma y) = \sigma f(y) \) whenever \( \sigma \in \text{Aut}(L'/K') \) and both \( y, \sigma y \in Y \), has a unique extension to a function \( f_1 \in C_{L|K}(X) \).

Proof. If \( x \in X \), take \( \sigma \in \text{Aut}(L'/K') \) such that \( \sigma x \in Y \) and define \( f_1(x) = \sigma^{-1}f(\sigma x) \). This definition is independent of the choice of \( \sigma \), and \( f_1 \) is the unique extension of \( f \) to \( X \) which satisfies \( f_1(\sigma x) = \sigma f_1(x) \) for all \( x \in X \) and \( \sigma \in \text{Aut}(L'/K') \). If \( f_1 \) were not continuous, there would exist a net \( x_i \in X \) converging to \( x_0 \in X \) such that the net \( f_1(x_i) \) would not converge to \( f_1(x_0) \). Suppose that \( x_i = \sigma_i y_i \) where \( \sigma_i \in \text{Aut}(L'/K') \) and \( y_i \in Y \). Since both \( \text{Aut}(L'/K') \) and \( Y \) are compact, we may assume, by taking subnets if necessary, that both \( \lim y_i = y_0 \) and \( \lim \sigma_i = \sigma_0 \) exist. Then \( \sigma_0 y_0 = x_0 \) and
\[
\lim f_1(x_i) = \lim \sigma_i f(y_i) = \sigma_0 f(y_0) = f_1(x_0).
\]
This contradiction shows that \( f_1 \) is continuous.

We now consider a special case of the above results, which is of interest in applications. Suppose that \( K \) is a finite algebraic extension of a field of \( p \)-adic numbers \( Q_p \) and that \( L = \overline{K} \) the algebraic closure of \( K \). We take \( X \) to be an invariant compact subset of \( \overline{K} \) (the action of \( \text{Aut}(\overline{K}/K) \) is the usual one) and note that the map of \( \text{Aut}(\overline{K}/K) \times X \rightarrow X \) given by \( (\sigma, x) \rightarrow \sigma x \) is continuous. In fact given \( \sigma_0 \in \text{Aut}(\overline{K}/K) \), \( x_0 \in X \), and \( \varepsilon > 0 \), put
\[
H = \{ \sigma \in \text{Aut}(\overline{K}/K) : \sigma x_0 = x_0 \}
\]
and
\[
N = \{ x \in X : |x - x_0| < \varepsilon \};
\]
then both \( H \) and \( N \) are open and \( HN = N \). We then obtain

THEOREM 8. Suppose \( I \) is an ideal of \( K[x] \); then the uniform closure of \( I \) in \( C_{\overline{K}|K}(X) \) is the set of functions \( f \in C_{\overline{K}|K}(X) \) which vanish at every zero of \( I \).
REFERENCES


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