Pacific Journal of Mathematics

ON THE STONE-WEIERSTRASS APPROXIMATION THEOREM FOR VALUED FIELDS

David Geoffrey Cantor

Vol. 21, No. 3 BadMonth 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 3, 1967

ON THE STONE-WEIERSTRASS APPROXIMATION THEOREM FOR VALUED FIELDS

DAVID G. CANTOR

Let X be a compact topological space, L a non-Archimedean rank 1 valued field and $\mathfrak{F}$ a uniformly closed L-algebra of L-valued continuous functions on X. Kaplansky has shown that if $\mathfrak{F}$ separates the points of X, then either $\mathfrak{F}$ consists of all L-valued continuous functions on X or else all of them which vanish on one point in X. In this paper analogous results are obtained, in the case that a group of transformations acts both on X and L, for the invariant L-valued continuous functions on X.

If L and K are fields such that $L \subset K$ and L/K is normal, we let $\mathrm{Aut}\,(L/K)$ denote the group of automorphisms of L which leave every element of K fixed, and we give $\mathrm{Aut}\,(L/K)$ the Krull topology; a basis for the open neighborhoods of the identity of $\mathrm{Aut}\,(L/K)$ is given by subgroups of the form

$$\{\sigma \in \mathrm{Aut}\,(L/K) : \sigma x = x \text{ if } x \in L_1\}$$

where L_1 is a finite extension of K contained in L.

Now suppose that L is a non-Archimedean field with a (multiplicative) rank 1 valuation, denoted $|\ \ |$ [1]. Suppose K is a subfield of L such that L/K is both normal and separable. Denote by L_c a completion of L and let K' be the closure of K in L_c. Put $L' = LK'$ (the composite field generated by L and K' in L_c) and note that K is dense in K'. It is clear that L'/K' is normal and separable. If $\sigma \in \mathrm{Aut}\,(L'/K')$, then, since K' is complete, $|\sigma x| = |x|$ for each $x \in L'$ so that σ is a continuous map of L' onto itself; furthermore the restriction of σ to L, $\sigma|_L \in \mathrm{Aut}\,(L/K)$. Finally suppose that X is a compact topological space for which there exists a continuous map $(\sigma, x) \to \sigma x$ of $\mathrm{Aut}\,(L'/K') \times X \to X$ satisfying $\sigma_1(\sigma_2 x) = (\sigma_1\sigma_2)x$ if $\sigma_1, \sigma_2 \in \mathrm{Aut}\,(L'/K')$, $x \in X$ and satisfying $ex = x$ if e is the identity of $\mathrm{Aut}\,(L'/K')$ and $x \in X$. It is immediate that if $\sigma \in \mathrm{Aut}\,(L'/K')$ then the map $x \to \sigma x$ of $X \to X$ is a homeomorphism of X. We shall call a set $Y \subset X$ *invariant* if $\mathrm{Aut}\,(L'/K')Y = Y$. Denote by $C_{L/K}(X)$ the set of L-valued continuous functions f on X satisfying $f(\sigma x) = \sigma f(x)$ for all $x \in X$ and $\sigma \in \mathrm{Aut}\,(L'/K')$; $C_{L/K}(X)$ is a K-algebra. If E is any valued field, denote by $C_E(X)$ the continuous E-valued functions on X and give $C_E(X)$ the sup-norm topology. Clearly $C_L(X) \supset C_{L/K}(X) \supset C_K(X)$.

THEOREM 1. *Suppose $\mathfrak{F}$ is a closed (in the sup-norm) K-sub-*

algebra of $C_{L/K}(X)$ *which separates the points of* X (i.e. *if* $x, y \in X$ *and* $x \neq y$, *there exists* $f \in \mathfrak{F}$ *such that* $f(x) \neq f(y)$). *Then either* $\mathfrak{F} = C_{L/K}(X)$ *or there exists* $x_0 \in X$ *such that*

$$\mathfrak{F} = \{f \in C_{L/K}(X) : f(x_0) = 0\} .$$

In the latter case the set $\{x_0\}$ *is invariant.*

Proof. Let $\mathfrak{F}'$ be the uniform closure of the K' algebra of functions generated by $\mathfrak{F}$ in $C_{L'}(X)$; since K is dense in K', $\mathfrak{F}$ is dense in $\mathfrak{F}'$ and hence it suffices to prove that $\mathfrak{F}' = C_{L'/K'}(X)$ or that $\mathfrak{F}' = \{f \in C_{L'/K'}(X) : f(x_0) = 0\}$. Thus we may assume without loss of generality that $K = K'$ and $L = L'$. We assume first that for each $x \in X$, there exists $f \in \mathfrak{F}$ such that $f(x) \neq 0$

LEMMA 2. *Assuming the hypotheses of Theorem* 1, *if* $x_0 \in X$ *and* $g \in C_{L/K}(X)$, *there exists* $f \in \mathfrak{F}$ *such that* $f(x_0) = g(x_0)$.

Proof. Put $L_1 = \{h(x_0) : h \in \mathfrak{F}\}$; clearly L_1 is a K-subalgebra of L containing a nonzero element of L. Suppose $c \in L_1$ and $c \neq 0$; c satisfies a polynomial equation $\sum_{i=0}^{n} a_i c^i = 0$, where the $a_i \in K$ and $a_0 \neq 0$. Then $a_0 \in L_1$ and hence $K = K a_0 \subset L_1$. It follows that L_1 is a subfield of L. Put

$$H = \{\sigma \in \operatorname{Aut}(L/K) : \sigma x_0 = x_0\} ;$$

H is a closed subgroup of $\operatorname{Aut}(L/K)$ which fixes every element of L_1 and also fixes $g(x_0)$. Now if $\sigma \in \operatorname{Aut}(L/K) - H$, then $x_0 \neq \sigma x_0$, and there exists $h \in \mathfrak{F}$ such that $h(x_0) \neq h(\sigma x_0)$ or $h(x_0) \neq \sigma h(x_0)$. Equivalently, if $\sigma \in \operatorname{Aut}(L/K)$ fixes every element of L_1, then $\sigma \in H$. Thus L_1 is the fixed field of the closed subgroup H. As H fixes $g(x_0)$, we have $g(x_0) \in L_1$, and there exists $f \in \mathfrak{F}$ such that $f(x_0) = g(x_0)$.

LEMMA 3. *Assuming the hypotheses of Theorem* 1, X *is totally disconnected.*

Proof. Since $\mathfrak{F}$ separates points, X is Hausdorff. Now take $x_0 \in X$ and an open neighborhood U of x_0. For each $y \notin U$, there exists $f_y \in \mathfrak{F}$ such that $f_y(x_0) \neq f_y(y)$. Put $\varepsilon_y = |f_y(x_0) - f_y(y)|$, and let

$$U_y = \{x \in X : |f_y(x) - f_y(x_0)| < \varepsilon_y/2\}$$

and

$$V_y = \{x \in X : |f_y(x) - f_y(y)| < \varepsilon_y/2\} ;$$

U_y and V_y are disjoint open and closed subsets of X with $x_0 \in U_y$.

The V_y cover the compact set $X - U$ and hence there exists a finite number, say $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$ whose union contains $X - U$. Then $\bigcap_{i=1}^{n} U_{y_i}$ is an open and closed neighborhood of x contained in U.

LEMMA 4. *Assuming the hypotheses of Theorem* 1, *suppose* V *is an open and closed invariant subset of* X. *Then the characteristic function of* V *is in* $\mathfrak{F}$.

Proof. By the Kaplansky-Stone-Weierstrass Theorem [2] and Lemma 3, the characteristic function of V is in the uniform closure of the L-subalgebra of $C_L(X)$ generated by $\mathfrak{F}$. Hence, if $\varepsilon > 0$, there exists $f \in C_L(X)$ such that $f = \sum_{i=1}^{m} a_i h_i$ where the $a_i \in L$ and the $h_i \in \mathfrak{F}$ and such that $|f(y) - 1| < \varepsilon$ if $y \in V$ while $|f(y)| < \varepsilon$ if $y \notin V$. Let $L_1 \subset L$ be the smallest normal extension field of K containing all of the a_i; L_1 is a finite algebraic extension of K and hence $\mathrm{Aut}(L_1/K)$ is finite. As $\mathrm{Aut}(L_1/K)$ is a homomorphic image of $\mathrm{Aut}(L/K)$, there exist representatives $\sigma_1, \sigma_2, \cdots, \sigma_n$ of $\mathrm{Aut}(L_1/K)$ in $\mathrm{Aut}(L/K)$ and the set of restrictions $\{\sigma_i|_{L_1} : 1 \leq i \leq n\}$ is $\mathrm{Aut}(L_1/K)$. If $\sigma \in \mathrm{Aut}(L/K)$, put $f^\sigma = \sum_{i=1}^{m} (\sigma a_i) h_i$. Then if $y \in X$,

$$f^\sigma(y) = \sum_{i=1}^{m} (\sigma a_i) h_i(\sigma\sigma^{-1} y)$$
$$= \sigma\Big(\sum_{i=1}^{m} a_i h_i(\sigma^{-1} y)\Big)$$
$$= \sigma f(\sigma^{-1} y) .$$

As $\sigma^{-1}V = V$, $|f^\sigma(y) - 1| < \varepsilon$ if $y \in V$, while $|f^\sigma(y)| < \varepsilon$ if $y \notin V$. Put $g = \prod_{i=1}^{n} f^{\sigma_i}$; then $g \in \mathfrak{F}$ and $|g(y) - 1| < \varepsilon$ if $y \in V$ while $|g(y)| < \varepsilon$ if $y \notin V$. Thus letting $\varepsilon \to 0$, we see that the characteristic function of V is in $\mathfrak{F}$.

Proof of Theorem 1 (concluded). Suppose $f \in C_{L/K}(X)$ and $\varepsilon > 0$. For each $x \in X$, there exists by Lemma 2, $g_x \in \mathfrak{F}$ such that $g_x(x) = f(x)$. Let U_x be an open and closed neighborhood of x such that $|g_x(y) - f(y)| < \varepsilon$ whenever $y \in U_x$. Put $V_x = \mathrm{Aut}(L/K)U_x$; clearly V_x is invariant. As V_x is the union of the open sets σU_x, $\sigma \in \mathrm{Aut}(L/K)$, V_x is open, and since it is the continuous image of the compact set $\mathrm{Aut}(L/K) \times U_x$, it is compact. If $y \in V_x$, there exists $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma y \in U_x$. Then

$$|g_x(y) - f(y)| = |\sigma(g_x(y) - f(y))|$$
$$= |g_x(\sigma y) - f(\sigma y)| < \varepsilon .$$

The V_x are open sets which cover X. Hence a finite number, say $V_{x_1}, V_{x_2}, \cdots, V_{x_n}$ cover X. Put $D_1 = V_{x_1}$ and for $2 \leq i \leq n$, put

$$D_i = V_{x_i} - \bigcup_{j=1}^{i-1} V_{x_j} .$$

Each D_i is open and closed, and invariant; hence by Lemma 4, the characteristic function h_i of D_i is in $\mathfrak{F}$. In addition the D_i are disjoint and $\bigcup_{i=1}^{n} D_i = X$. Now put

$$g = \sum_{i=1}^{n} h_i g_{x_i} ,$$

so that $g \in \mathfrak{F}$. If $y \in X$, then there exists j such that $y \in D_j \subset V_{x_j}$; then $g(y) = g_{x_j}(y)$. As $|g_{x_j}(y) - f(y)| < \varepsilon$, $|g(y) - f(y)| < \varepsilon$. Letting $\varepsilon \to 0$ shows that $f \in \mathfrak{F}$. Finally, if there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathfrak{F}$, let $\mathfrak{F}_1$ be the K-algebra obtained from $\mathfrak{F}$ by adjoining the K-valued constant functions. Then if $g \in C_{L/K}(X)$ satisfies $g(x_0) = 0$, and $\varepsilon > 0$, there exists by what we have proved $f_1 \in \mathfrak{F}_1$ such that $|f_1(x) - f(x)| < \varepsilon$ for all $x \in X$. Then $f_1 = f + a$, where $f \in \mathfrak{F}$ and $a \in K$. Now $|a| = |f_1(x_0)| < \varepsilon$, hence $|f(x) - g(x)| < \varepsilon$ for all $x \in X$. Letting $\varepsilon \to 0$ shows that $g \in \mathfrak{F}$.

COROLLARY 5. *Suppose that $C_{L/K}(X)$ separates the points of X and that I is a closed ideal of the K-algebra $C_{L/K}(X)$. Then there exists a closed invariant set $Y \subset X$ such that*

$$I = \{f \in C_{L/K}(X) : f(Y) = \{0\}\} .$$

Proof. Put $Y = \bigcap_{f \in I} \{x : f(x) = 0\}$. Then Y is a closed invariant subset of X. If $x_1, x_2 \in X - Y$ and $x_1 \neq x_2$, then there exists $f \in I$ such that $f(x_1) \neq 0$. If $f(x_1) \neq f(x_2)$, let g be the constant function 1, while if $f(x_1) = f(x_2)$, choose $g \in C_{L/K}(X)$ such that $g(x_1) \neq g(x_2)$. Then in either case the function $h = gf \in I$ and $h(x_1) \neq h(x_2)$. Now let X_1 be the topological space obtained from X by identifying the points of Y, and let p be the projection from X to X_1. Then p is continuous and if $x_1, x_2 \in X$, we have $p(x_1) = p(x_2)$ if and only if either $x_1 = x_2$ or $x_1, x_2 \in Y$. A basis for the open neighborhoods of a point $x \in X_1$ is given by sets of the form $p(V)$, where V is an open neighborhood of $p^{-1}(x)$ in X. If $\sigma \in \mathrm{Aut}(L'/K')$ and $x \in X_1$, we define $\sigma x = p(\sigma p^{-1}(x))$; this is well defined and yields a continuous map $(\sigma, x) \to \sigma x$ of $\mathrm{Aut}(L'/K') \times X_1 \to X_1$. Denote by $C_{L/K}(X, Y)$ the K-algebra of $f \in C_{L/K}(X)$ which are constant on Y. If $f \in C_{L/K}(X, Y)$ define $pf \in C_{L/K}(X_1)$ by $(pf)(x) = f(p^{-1}(x))$; this is well defined and yields a norm preserving isomorphism between $C_{L/K}(X, Y)$ and $C_{L/K}(X_1)$. Put $pI = \{pf : f \in I\}$; pI is a uniformly closed K-subalgebra which separates the points of X_1, and every function $pf \in pI$ vanishes on $p(Y)$; hence by Theorem 1, pI consists of all $f \in C_{L/K}(X_1)$ which vanish on $p(Y)$. Thus I consists of all $f \in C_{L/K}(X)$ which vanish on Y.

COROLLARY 6. *Suppose that $C_{L/K}(X)$ separates the points of X. Then the maximal ideals of the K-algebra $C_{L/K}(X)$ are precisely the sets of the form*

$$\{f \in C_{L/K}(X) : f(x_0) = 0\}$$

where $x_0 \in X$.

The following theorem permits the extension of Theorem 1 and its corollaries to certain subsets of X.

THEOREM 7. *Suppose Y is a closed subset of X and* $\mathrm{Aut}(L'/K')Y = X$. *Then each continuous K-valued function f on Y, satisfying $f(\sigma y) = \sigma f(y)$ whenever $\sigma \in \mathrm{Aut}(L'/K')$ and both $y, \sigma y \in Y$, has a unique extension to a function $f_1 \in C_{L/K}(X)$.*

Proof. If $x \in X$, take $\sigma \in \mathrm{Aut}(L'/K')$ such that $\sigma x \in Y$ and define $f_1(x) = \sigma^{-1}f(\sigma x)$. This definition is independent of the choice of σ, and f_1 is the unique extension of f to X which satisfies $f_1(\sigma x) = \sigma f_1(x)$ for all $x \in X$ and $\sigma \in \mathrm{Aut}(L'/K')$. If f_1 were not continuous, there would exist a net $x_i \in X$ converging to $x_0 \in X$ such that the net $f_1(x_i)$ would not converge to $f_1(x_0)$. Suppose that $x_i = \sigma_i y_i$ where $\sigma_i \in \mathrm{Aut}(L'/K')$ and $y_i \in Y$. Since both $\mathrm{Aut}(L'/K')$ and Y are compact, we may assume, by taking subnets if necessary, that both $\lim y_i = y_0$ and $\lim \sigma_i = \sigma_0$ exist. Then $\sigma_0 y_0 = x_0$ and

$$\lim f_1(x_i) = \lim \sigma_i f(y_i) = \sigma_0 f(y_0) = f_1(x_0) .$$

This contradiction shows that f_1 is continuous.

We now consider a special case of the above results, which is of interest in applications. Suppose that K is a finite algebraic extension of a field of p-adic numbers Q_p and that $L = \tilde{K}$ the algebraic closure of K. We take X to be an invariant compact subset of $\tilde{K}$ (the action of $\mathrm{Aut}(\tilde{K}/K)$ is the usual one) and note that the map of $\mathrm{Aut}(\tilde{K}/K) \times X \rightarrow X$ given by $(\sigma, x) \rightarrow \sigma x$ is continuous. In fact given $\sigma_0 \in \mathrm{Aut}(\tilde{K}/K)$, $x_0 \in X$, and $\varepsilon > 0$, put

$$H = \{\sigma \in \mathrm{Aut}(\tilde{K}/K) : \sigma x_0 = \sigma_0 x_0\}$$

and

$$N = \{x \in X : |x - x_0| < \varepsilon\} ;$$

then both H and N are open and $HN = N$. We then obtain

THEOREM 8. *Suppose I is an ideal of $K[x]$; then the uniform closure of I in $C_{\tilde{K}/K}(X)$ is the set of functions $f \in C_{\tilde{K}/K}(X)$ which vanish at every zero of I.*

REFERENCES

1. E. Artin, *Theory of Algebraic Numbers,* Göttingen, 1959.
2. I. Kaplansky, *The Weierstrass Theorem in fields with valuations,* Proc. Amer. Math. Soc. **1** (1950), 356-357.

Received July 11, 1966. The preparation of this paper was sponsored in part by N.S.F. Grant GP 5497.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 21, No. 3 BadMonth, 1967