

Pacific Journal of Mathematics

STABILITY IN TOPOLOGICAL DYNAMICS

JAMES WALTON ENGLAND

STABILITY IN TOPOLOGICAL DYNAMICS

JAMES W. ENGLAND

This paper is concerned with two types of stability in transformation groups. The first is a generalization of Lyapunov stability. In the past this notion has been discussed in a setting where the phase group was either the integers or the one-parameter group of reals. In this paper it is defined for replete subsets of a more general phase group in a transformation group. Some connections between this type of stability and almost periodicity are given. In particular, it is shown that a type of uniform Lyapunov stability will imply Bohr almost periodicity. The second type of stability in this paper is a limit stability. This gives a condition which is necessary and sufficient for the limit set to be a minimal set. Finally, these two types of stability are combined to provide a sufficient condition for a limit set to be the closure of a Bohr almost periodic orbit.

Throughout this paper X will be assumed to be a uniform space. It will be implicitly assumed that the Hausdorff topology of X is the one induced by the uniformity. T will denote a topological group and the triple (X, T, π) will be called a transformation group provided X and T are as above and $\pi: X \times T \rightarrow X$ such that if e is the identity of T then:

- (1) $\pi(x, e) = x$ for all x in X ,
- (2) $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 t_2)$ for all x in X and t_1, t_2 in T ,
- (3) π is continuous. Henceforth we shall write $\pi(x, t) = xt$; and if $A \subset T$ then $xA = \{xt: t \in A\}$.

DEFINITION 1. A subset A of T is called *left* *right* *syndetic* [6] in T provided there exists a compact set $K \subset T$ such that $\{AK = T\}$ $\{KA = T\}$. It is clear that if A is left syndetic in T then A^{-1} is right syndetic in T .

DEFINITION 2. A point $x \in X$ is called *S-Lyapunov stable* ($S \subset T$) with respect to a set $B \subset X$ provided that for each index α of X there exists an index β of X such that if $y \in B \cap x\beta$ then $yt \in x\alpha$ for all t in S .

THEOREM 1. *If S is left syndetic in T and $\text{Cl}(xT)$ ($=$ closure xT) is compact then a necessary and sufficient condition that $x \in X$ be T -Lyapunov stable with respect to xT is that x be S -Lyapunov stable with respect to xT .*

Proof. The necessity is clear. To prove the sufficiency let α , an index of X , be given. Since S is left syndetic there exists a compact set $K \subset T$ such that $SK = T$. Since both $\text{Cl}(xT)$ and K are compact the mapping $\pi: \text{Cl}(xT) \times K \rightarrow \text{Cl}(xT)$ is uniformly continuous. Hence there exists an index β of X such that if $p \in \text{Cl}(xT)$ and $q \in xT$ such that $q \in p\beta$ then $qk \in (pk)\alpha$ for all $k \in K$. The assumption that x is S -Lyapunov stable with respect to xT implies that there exists an index γ of X such that if $y \in x\gamma \cap xT$ then $ys \in (xs)\beta$ for all $s \in S$. Let $t \in T$ and $y \in x\gamma \cap xT$ be given. There exists an $s \in S$ and $k \in K$ such that $sk = t$. Thus $ys \in (xs)\beta$ which implies that $y(sk) \in (x(sk))\alpha$ or $yt \in x\alpha$. Since t is arbitrary the theorem is proved.

Simple examples show that in one sense this is about as strong an inheritance theorem that one can prove. For example, if $\text{Cl}(xT)$ is not compact then S being syndetic in T and x being S -Lyapunov stable is not sufficient for x to be T -Lyapunov stable.

We now consider some connections between S -Lyapunov stability and almost periodicity. To do this we need the following lemma which provides a characterization of repleteness.

DEFINITION 3. A subset M of T is said to be *replete* [6] in T provided M contains some bilateral translate of each compact subset of T .

LEMMA 1. *In order that a subset S of T be replete it is sufficient that S intersect each translate of each left syndetic subset of T and if T is commutative this condition is also necessary.*

Proof. In order to show that this condition is sufficient we assume that S is not replete in T . That is, there exists a compact set $K \subset T$ such that for all $t_1, t_2 \in T$ we have $t_1 K t_2 \not\subset S$. Let $A(t_1, t_2) = t_1 K t_2 - S$ and $A = \bigcup A(t_1, t_2)$, $((t_1, t_2) \in T \times T)$. It follows that $A = T - S$. Clearly we can assume that $e \in K$. We now show that A is left syndetic. Since $e \in K^{-1}$ it follows that $T - S \subset AK^{-1}$. Let $s \in S$ be given. If $sK \cap A = \emptyset$ then $sKe \subset S$ since $A = T - S$, which is impossible. Therefore $sK \cap A \neq \emptyset$. Hence there exists a $k \in K$ and $a \in A$ such that $sk = a$. Therefore $s = ak^{-1}$ and $s \in AK^{-1}$ which, since s was arbitrary, implies that $S \subset AK^{-1}$. Therefore $AK^{-1} = T$ and A is left syndetic in T . However $Ae \cap S = \emptyset$ and this contradiction proves the sufficiency.

To show that this condition is also necessary if T is commutative we assume that S is replete and that there exists a syndetic set $A \subset T$ and a $t' \in T$ such that $t'A \cap S = \emptyset$. Let K be a compact set which contains the identity and has the property that $AK = T$. Since

S is replete there exists a $t_1 \in T$ such that $K^{-1}t_1 \subset S$. (Repleteness reduces to this property when T is commutative.) Since $e \in K^{-1}$ it follows that $t_1 \in S$. Also, $AK = T$ which implies that $t'AK = T$. Hence there exists an $a \in A$ and $k \in K$ such that $t'ak = t_1$. This implies that $t'a = t_1k^{-1}$. Hence $t'a \in S$ and $t'A \cap S \neq \emptyset$ which is a contradiction.

DEFINITION 4. T is said to be *almost periodic at x* [6] provided that for each index α of X there exists a left syndetic set $A \subset T$ such that $xA \subset x\alpha$.

DEFINITION 5. T is said to be *Bohr almost periodic at $x \in X$* (x is *Bohr almost periodic*) provided that corresponding to each index α of X there exists a left syndetic set A in T such that $xtA \subset x\alpha$ for all $t \in T$.

It is clear that if T is commutative and x is both almost periodic and T -Lyapunov stable with respect to xT then x is Bohr almost periodic. However, it is possible to weaken these conditions and still obtain Bohr almost periodicity. Throughout the rest of this paper it will be assumed that T is commutative.

THEOREM 2. *Let S be a replete subset of T . If $x \in X$ is S -Lyapunov stable with respect to xT and x is almost periodic then x is Bohr almost periodic.*

Proof. Let α , an index of X , be given and let β be a symmetric index of X such that $\beta^2 \subset \alpha$. Let S be a replete subset of T such that x is S -Lyapunov stable with respect to xT . Then there exists an index γ of X such that if $y \in x\gamma \cap xT$ then $yt \in (xt)\beta$ for all t in S . Let δ be a symmetric index of X such that $\delta^2 \subset \gamma$. Since x is almost periodic under T there exists a syndetic set $A \subset T$ such that $xA \subset x\delta$. Let $t' \in T$ and $a \in A$ be given. We will now show that $x(t'a) \in x\alpha$ which will complete the proof. Since π is continuous there exists an index σ of X such that if $y \in x\sigma$ then $ya \in x\delta$. Let η be an index of X with the property that $\eta \subset \sigma \cap \delta$. Once again there exists a syndetic set $B \subset T$ such that $xB \subset x\eta$ since x is almost periodic. Since S is a replete subset of T , S^{-1} is also replete in T . Also, since Bt'^{-1} is syndetic it follows from Lemma 1 that $Bt'^{-1} \cap S^{-1} \neq \emptyset$. That is, there exists an $s \in S$ and $t_1 \in B$ such that $t_1t'^{-1} = s^{-1}$ which implies that $xt_1 \in x\eta$ thus $xt_1a \in x\delta$. Hence $xt_1a \in x\alpha$. The fact that $xa \in x\delta$ implies that $xt_1a \in x\gamma$. Since $t't_1^{-1} = s \in S$ it follows that

$$xt_1at't_1^{-1} \in (xt't_1^{-1})\beta$$

or

$$x(t'a) \in x(t't_1^{-1})\beta$$

However since $xt_1 \in x\gamma$ it follows that $xt_1 \in \delta$. Therefore

$$xt_1(t't_1^{-1}) \in (xt't_1^{-1})\beta$$

or $xt' \in (xt't_1^{-1})\beta$. Since β is symmetric and $\beta^2 \subset \alpha$ it follows that $xt'a \in xt'\alpha$ which completes the proof.

In [6, 6.34] the *P-limit set* of x for $P \subset T$ and $x \in X$ is defined by $P_x = \bigcap \text{Cl}(xtP)(t \in T)$. In this same reference it is stated that if P is a replete semi-group in T then P_x is closed and invariant and if $\text{Cl}(xP)$ is compact then $P_x \neq \emptyset$. Using this notion it is possible to give another set of conditions which are sufficient for x to be Bohr almost periodic. This theorem generalizes a theorem of A. A. Markov [7, p. 390].

DEFINITION 6. The orbit of a point $x \in X$ is said to be (*uniformly*) *S-Lyapunov stable* with respect to a set $B \subset X$ provided that for each index α of X there exists an index β of X such that if $y \in xT$ and $z \in B$ with $y \in z\beta$ then $yt \in zt\alpha$ for all t in S .

THEOREM 3. Let S be a replete semi-group in T . If the orbit of x is *S-Lyapunov stable* with respect to xT and $\text{Cl}(xS^{-1})$ is compact then x is Bohr almost periodic.

Proof. If we can show under these hypotheses that x is almost periodic under T then by using Theorem 2 we can deduce that x is Bohr almost periodic.

Let S be a replete semi-group of T , let the orbit of x be *S-Lyapunov stable* with respect to xT and let $\text{Cl}(xS^{-1})$ be compact. It is clear that S^{-1} is also a replete semi-group of T . Therefore, by the above remarks it follows that S_x^{-1} is nonempty. Since S_x^{-1} is closed it is compact. It follows from [6, 4.06] that there exists a $y \in S_x^{-1}$ such that y is almost periodic. It follows from [6, 4.07] that $\text{Cl}(yT)$ is a compact minimal set. If $x \in \text{Cl}(yT)$ then x is almost periodic and the theorem is proved.

Assume $x \notin \text{Cl}(yT)$. Then there exists an index α of X with the property that $x \notin (\text{Cl}(yT))\alpha$. Let β be an index of X with the property that $\beta^2 \subset \alpha$. Since the orbit of x is *S-Lyapunov stable* with respect to xT there exists an index γ of X with the property that if $p \in q\gamma$ and $p, q \in xT$ then $pt \in qt\beta$ for all $t \in S$. Let δ be a symmetric index of X with the property that $\delta^2 \subset \gamma$. Since $y \in S_x^{-1} - xT$ and $S_x^{-1} = \bigcap_{t \in T} \text{Cl}(xtS^{-1})$ we have $y \in S_x^{-1} \subset \text{Cl}(xS^{-1})$. Hence there exists an $s_1 \in S$ such that $xs_1^{-1} \in y\delta$. Since π is continuous there exists an index σ of X such that if $p \in y\sigma$ then $ps_1 \in ys_1\beta$. There exists an $s \in S$ such that $xs^{-1} \in y\sigma \cap y\delta$. Thus $xs^{-1}s_1 \in ys_1\beta$. Also $xs \in y\delta$ which implies that $xs_1^{-1} \in xs^{-1}\gamma$. Thus $xs_1^{-1}s_1 \in xs^{-1}s_1\beta$. These two statements imply that

$$x = x s_1^{-1} s_1 \in x s_1 \beta^2 \subset (\text{Cl}(xT)) \alpha$$

which is a contradiction. It follows that $x \in \text{Cl}(yT)$ and the theorem is proved.

A subset E of T is said to be *P-extensive* (P is a replete semi-group not equal to T) [2, p. 1146] provided that $pP \cap E \neq \emptyset$ for all p in P . A point $x \in X$ is said to be *P-recurrent* provided that for each index α of X there exists a P -extensive set E such that $xE \subset x\alpha$. Using these concepts and the previous theorem we are able to give a set of necessary and sufficient conditions in order for a point to be Bohr almost periodic.

THEOREM 4. *If S_x is compact for some replete semigroup S of T then the following statements are equivalent:*

- (1) *x is Bohr almost periodic,*
- (2) *x is S -recurrent and the orbit of x is S -Lyapunov stable with respect to S_x .*

Proof. If x is Bohr almost periodic and α is any index of X then there exists a syndetic set $A \subset T$ such that $xtA \subset (xt)\alpha$ for all t in T . It follows from Lemma 1 that A is P -extensive for each replete semi-group of T . Hence x is S -recurrent. From [1] it follows that $\text{Cl}(xT) = S_x$ and hence $\text{Cl}(xT)$ is compact. By [6, 4.37] it follows that the orbit of x is T -Lyapunov stable with respect to xT . It follows in the same manner as in [7, p. 385] that the orbit of x is T -Lyapunov stable with respect to $\text{Cl}(xt)$. Since $S_x = \text{Cl}(xT)$ the orbit of x is T -Lyapunov stable with respect to S_x which completes the proof of this half of the theorem.

If x is S -recurrent then it follows that $\text{Cl}(xT) = S_x$. Hence $\text{Cl}(xT)$ is compact. Since the orbit of x is S -Lyapunov stable with respect to S_x it follows from Theorem 3 that x is Bohr almost periodic.

An alternate proof of this theorem can be given using the main theorem in [3] and the theorem of Gottschalk [5] relating uniform almost periodicity and equi-continuity.

We now introduce the concept of S -orbital stability which is generalized from the notion of a final point being asymptotically stable which was discussed by Friedlander [4].

DEFINITION 6. A point $y \in X$ is said to be *S-orbitally stable* (*SO-stable*) with respect to a set $B \subset X$ provided there exists an open set U containing y such that if $x \in B \cap U$ then $S_x = S_y$. When X is a uniform space then the orbit of y is *uniformly SO-stable* with respect to $B \subset X$ provided there exists an index α of X such that if $x \in B \cap y\alpha$ then $S_x = S_y$.

LEMMA 2. *If S is a replete semi-group and S_y is compact and nonempty then a necessary and sufficient condition that S_y be a minimal set is that the orbit of y be uniformly SO -stable with respect to S_y .*

Proof. If S_y is a minimal set then it follows immediately that the orbit of y is uniformly SO -stable with respect to S_y .

Let the orbit of y be uniformly SO -stable with respect to S_y and $x \in S_y$. There exists an index δ of X such that if $z \in S_y \cap yt\delta$ then $S_z = S_y$. Since $x \in S_y$ there exists a $t' \in T$ such that $x \in yt'\delta$ which implies $S_x = S_{y'}$. But, since S is a replete semi-group of T , $S_{y'} = S_y$ hence $S_x = S_y$. Therefore, $S_x \subset \text{Cl}(xS) \subset \text{Cl}(xT)$ implies $S_y = \text{Cl}(xT)$ for all $x \in S_y$. Thus S_y is a minimal set.

THEOREM 5. *Let S be a replete semi-group in T and let S_y be compact and nonempty. If the orbit of y is uniformly SO -stable and S -Lyapunov stable with respect to S_y then S_y is the closure of a Bohr almost periodic orbit.*

Proof. It follows from the previous lemma that S_y is a minimal set. By Theorem 4 it is sufficient to show that if $x \in S_y$ then the orbit of x is S -Lyapunov stable with respect to $S_y = S_x$. Let δ be an index of X and β be a symmetric index of X such that $\beta^2 \subset \delta$. Since the orbit of y is S -Lyapunov stable with respect to S_y there exists an index γ of X such that if $z \in S_y \cap (yt')\gamma$ then $zt \in (yt't)\beta$ for all $t \in S$. Let α be a symmetric index of X with the property that $\alpha^2 \subset \gamma$. Let $z \in S_y \cap (xt')\alpha$. There exists a $\bar{t} \in T$ such that $y\bar{t} \in (xt')\alpha$ which implies $y\bar{t}t \in (xt't)\beta$ and $zt \in (y\bar{t}t)\beta$ for all $t \in S$. Hence $zt \in (xt't)\beta^2 \subset (xt't)\delta$ for all $t \in S$. This implies that x is S -Lyapunov stable with respect to S_y and completes the proof of the theorem.

The question of necessary and sufficient conditions on y in order that S_y be the closure of a Bohr almost periodic point is still an open question. Theorem 5 shows that a necessary condition must be found on y which will imply that $x \in S_y$ is uniformly S -Lyapunov stable with respect to S_y .

BIBLIOGRAPHY

1. J. D. Baum, *Asymptoticity in topological Dynamics*, Trans. Amer. Math. Soc. **77** (1954), 506-519.
2. ———, *P-recurrence in topological dynamics*, Proc. Amer. Math. Soc. **7** (1956), 1146-1154.
3. ———, *An equicontinuity condition in topological dynamics*, Proc. Amer. Math. Soc. **12** (1961), 30-32.
4. F. G. Friedlander, *On the iteration of a continuous mapping of a compact metric space into itself*, Proc. Cambridge Phil. Soc. **46** (1950), 46-56.

5. W. H. Gottschalk, *Almost periodicity, equi-continuity and total Boundedness*, Bull. Amer. Math. Soc. **52** (1946), 633-636.
6. W. H. Gottschalk, and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Coll. Pub. **36** (1955), New York.
7. V. V. Nemytskii and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Math. Series **22** (1960) Princeton, New Jersey.

Received February 21, 1966, and in revised November 8, 1966

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California

J. P. JANS
University of Washington
Seattle, Washington 98105

J. DUGUNDJI
University of Southern California
Los Angeles, California 90007

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 21, No. 3

BadMonth, 1967

Richard Allen Askey, <i>A transplantation theorem for Jacobi coefficients</i>	393
Raymond Balbes, <i>Projective and injective distributive lattices</i>	405
Raymond Balbes and Alfred Horn, <i>Order sums of distributive lattices</i>	421
Donald Charles Benson, <i>Nonconstant locally recurrent functions</i>	437
Allen Richard Bernstein, <i>Invariant subspaces of polynomially compact operators on Banach space</i>	445
Robert F. Brown, <i>Fixed points and fibre</i>	465
David Geoffrey Cantor, <i>On the Stone-Weierstrass approximation theorem for valued fields</i>	473
James Walton England, <i>Stability in topological dynamics</i>	479
Alessandro Figà-Talamanca and Daniel Rider, <i>A theorem on random Fourier series on noncommutative groups</i>	487
Sav Roman Harasymiv, <i>A note of dilations in L^p</i>	493
J. G. Kalbfleisch, <i>A uniqueness theorem for edge-chromatic graphs</i>	503
Richard Paul Kelisky and Theodore Joseph Rivlin, <i>Iterates of Bernstein polynomials</i>	511
D. G. Larman, <i>On the union of two starshaped sets</i>	521
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, <i>Properties of differential forms in n real variables</i>	525
John W. Moon and Leo Moser, <i>Generating oriented graphs by means of team comparisons</i>	531
Veikko Nevanlinna, <i>A refinement of Selberg's asymptotic equation</i>	537
Ulrich Oberst, <i>Relative satellites and derived functors of functors with additive domain</i>	541
John Vincent Ryff, <i>On Muirhead's theorem</i>	567
Carroll O. Wilde and Klaus G. Witz, <i>Invariant means and the Stone-Čech compactification</i>	577