STABILITY IN TOPOLOGICAL DYNAMICS

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This paper is concerned with two types of stability in transformation groups. The first is a generalization of Lyapunov stability. In the past this notion has been discussed in a setting where the phase group was either the integers or the one-parameter group of reals. In this paper it is defined for replete subsets of a more general phase group in a transformation group. Some connections between this type of stability and almost periodicity are given. In particular, it is shown that a type of uniform Lyapunov stability will imply Bohr almost periodicity. The second type of stability in this paper is a limit stability. This gives a condition which is necessary and sufficient for the limit set to be a minimal set. Finally, these two types of stability are combined to provide a sufficient condition for a limit set to be the closure of a Bohr almost periodic orbit.

Throughout this paper $X$ will be assumed to be a uniform space. It will be implicitly assumed that the Hausdorff topology of $X$ is the one induced by the uniformity. $T$ will denote a topological group and the triple $(X, T, \pi)$ will be called a transformation group provided $X$ and $T$ are as above and $\pi: X \times T \to X$ such that if $e$ is the identity of $T$ then:

1. $\pi (x, e) = x$ for all $x$ in $X$,
2. $\pi (\pi (x, t_1), t_2) = \pi (x, t_1 t_2)$ for all $x$ in $X$ and $t_1, t_2$ in $T$,
3. $\pi$ is continuous. Henceforth we shall write $\pi (x, t) = xt$; and if $A \subset T$ then $xA = \{xt: t \in A\}$.

DEFINITION 1. A subset $A$ of $T$ is called \{left\}{right} syndetic \[6\] in $T$ provided there exists a compact set $K \subset T$ such that $AK = T$, $KA = T$. It is clear that if $A$ is left syndetic in $T$ then $A^{-1}$ is right syndetic in $T$.

DEFINITION 2. A point $x \in X$ is called $S$-Lyapunov stable ($S \subset T$) with respect to a set $B \subset X$ provided that for each index $\alpha$ of $X$ there exists an index $\beta$ of $X$ such that if $y \in B \cap x\beta$ then $yt \in xt\alpha$ for all $t$ in $S$.

THEOREM 1. If $S$ is left syndetic in $T$ and $\text{Cl} (xT)$ ( = closure of $xT$) is compact then a necessary and sufficient condition that $x \in X$ be $T$-Lyapunov stable with respect to $xT$ is that $x$ be $S$-Lyapunov stable with respect to $xT$. 

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Proof. The necessity is clear. To prove the sufficiency let $\alpha$, an index of $X$, be given. Since $S$ is left syndetic there exists a compact set $K \subset T$ such that $SK = T$. Since both $\text{Cl}(xT)$ and $K$ are compact the mapping $\pi: \text{Cl}(xT) \times K \to \text{Cl}(xT)$ is uniformly continuous. Hence there exists an index $\beta$ of $X$ such that if $p \in \text{Cl}(xT)$ and $q \in xT$ such that $q \in p\beta$ then $qk \in (pk)\alpha$ for all $k \in K$. The assumption that $x$ is $S$-Lyapunov stable with respect to $xT$ implies that there exists an index $\gamma$ of $X$ such that if $y \in x\gamma \cap xT$ then $ys \in (xs)\beta$ for all $s \in S$. Let $t \in T$ and $y \in x\gamma \cap xT$ be given. There exists an $s \in S$ and $k \in K$ such that $sk = t$. Thus $ys \in (xs)\beta$ which implies that $y(sk) \in (x(sk))\alpha$ or $yt \in xt\alpha$. Since $t$ is arbitrary the theorem is proved.

Simple examples show that in one sense this is about as strong an inheritance theorem that one can prove. For example, if $\text{Cl}(xT)$ is not compact then $S$ being syndetic in $T$ and $x$ being $S$-Lyapunov stable is not sufficient for $x$ to be $T$-Lyapunov stable.

We now consider some connections between $S$-Lyapunov stability and almost periodicity. To do this we need the following lemma which provides a characterization of repleteness.

**Definition 3.** A subset $M$ of $T$ is said to be replete [6] in $T$ provided $M$ contains some bilateral translate of each compact subset of $T$.

**Lemma 1.** In order that a subset $S$ of $T$ be replete it is sufficient that $S$ intersect each translate of each left syndetic subset of $T$ and if $T$ is commutative this condition is also necessary.

**Proof.** In order to show that this condition is sufficient we assume that $S$ is not replete in $T$. That is, there exists a compact set $K \subset T$ such that for all $t_1, t_2 \in T$ we have $t_1Kt_2 \not\subset S$. Let $A(t_1, t_2) = t_1Kt_2 - S$ and $A = \bigcup A(t_1, t_2)$, $((t_1, t_2) \in T \times T)$. It follows that $A = T - S$. Clearly we can assume that $e \in K$. We now show that $A$ is left syndetic. Since $e \in K^{-1}$ it follows that $T - S \subset AK^{-1}$. Let $s \in S$ be given. If $sK \cap A = \emptyset$ then $sK \subset S$ since $A = T - S$, which is impossible. Therefore $sK \cap A \neq \emptyset$. Hence there exists a $k \in K$ and $a \in A$ such that $sk = a$. Therefore $s = ak^{-1}$ and $s \in AK^{-1}$ which, since $s$ was arbitrary, implies that $S \subset AK^{-1}$. Therefore $AK^{-1} = T$ and $A$ is left syndetic in $T$. However $Ae \cap S = \emptyset$ and this contradiction proves the sufficiency.

To show that this condition is also necessary if $T$ is commutative we assume that $S$ is replete and that there exists a syndetic set $A \subset T$ and a $t' \in T$ such that $t'A \cap S = \emptyset$. Let $K$ be a compact set which contains the identity and has the property that $AK = T$. Since
S is replete there exists a $t_i \in T$ such that $K^{-1} t_i \subset S$. (Repleteness reduces to this property when $T$ is commutative.) Since $e \in K^{-1}$ it follows that $t_i \in S$. Also, $AK = T$ which implies that $t'AK = T$. Hence there exists an $a \in A$ and $k \in K$ such that $t'ak = t_i$. This implies that $t'a = t_i k^{-1}$. Hence $t' \in S$ and $t'A \cap S \neq \emptyset$ which is a contradiction.

**Definition 4.** $T$ is said to be **almost periodic at** $x$ [6] provided that for each index $\alpha$ of $X$ there exists a left syndetic set $A \subset T$ such that $x A \subset x \alpha$.

**Definition 5.** $T$ is said to be **Bohr almost periodic at** $x \in X$ ($x$ is Bohr almost periodic) provided that corresponding to each index $\alpha$ of $X$ there exists a left syndetic set $A$ in $T$ such that $x t A \subset x t \alpha$ for all $t \in T$.

It is clear that if $T$ is commutative and $x$ is both almost periodic and $T$-Lyapunov stable with respect to $x T$ then $x$ is Bohr almost periodic. However, it is possible to weaken these conditions and still obtain Bohr almost periodicity. Throughout the rest of this paper it will be assumed that $T$ is commutative.

**Theorem 2.** Let $S$ be a replete subset of $T$. If $x \in X$ is $S$-Lyapunov stable with respect to $x T$ and $x$ is almost periodic then $x$ is Bohr almost periodic.

**Proof.** Let $\alpha$, an index of $X$, be given and let $\beta$ be a symmetric index of $X$ such that $\beta^2 \subset \alpha$. Let $S$ be a replete subset of $T$ such that $x$ is $S$-Lyapunov stable with respect to $x T$. Then there exists an index $\gamma$ of $X$ such that if $y \in x \gamma \cap x T$ then $yt \in (xt) \beta$ for all $t$ in $S$. Let $\delta$ be a symmetric index of $X$ such that $\delta^2 \subset \gamma$. Since $x$ is almost periodic under $T$ there exists a syndetic set $A \subset T$ such that $x A \subset x \delta$. Let $t' \in T$ and $a \in A$ be given. We will now show that $x(t'a) \in x t' \alpha$ which will complete the proof. Since $\pi$ is continuous there exists an index $\sigma$ of $X$ such that if $y \in x \sigma$ then $ya \in x a \delta$. Let $\eta$ be an index of $X$ with the property that $\eta \subset \sigma \cap \delta$. Once again there exists a syndetic set $B \subset T$ such that $xB \subset x \eta$ since $x$ is almost periodic. Since $S$ is a replete subset of $T$, $S^{-1}$ is also replete in $T$. Also, since $B t'^{-1}$ is syndetic it follows from Lemma 1 that $B t'^{-1} \cap S^{-1} \neq \emptyset$. That is, there exists an $s \in S$ and $t_i \in B$ such that $t_i t'^{-1} = s^{-1}$ which implies that $x t_i \in x \eta$ thus $x t_i \in x \sigma$. Hence $x t_i a \in x a \delta$. The fact that $xa \in x \delta$ implies that $x t_i a \in x \gamma$. Since $t't'^{-1} = s \in S$ it follows that $x t_i a t't'^{-1} \in (xt' t'^{-1}) \beta$

or

$x(t'a) \in x(t't'^{-1}) \beta$
However since $xt, \in \eta$ it follows that $xt, \in \delta$. Therefore
\[ xt,(t't) \in (xt't')\beta \]
or $xt' \in (xt't')\beta$. Since $\beta$ is symmetric and $\beta^2 \subset \alpha$ it follows that $xt'a \in xt'\alpha$ which completes the proof.

In [6, 6.34] the $P$-limit set of $x$ for $P \subset T$ and $x \in X$ is defined by $P_x = \bigcap \text{Cl}(xtP)(t \in T)$. In this same reference it is stated that if $P$ is a replete semi-group in $T$ then $P_x$ is closed and invariant and if $\text{Cl}(xP)$ is compact then $P_x \neq \emptyset$. Using this notion it is possible to give another set of conditions which are sufficient for $x$ to be Bohr almost periodic. This theorem generalizes a theorem of A. A. Markov [7, p. 390].

**Definition 6.** The orbit of a point $x \in X$ is said to be (uniformly) $S$-Lyapunov stable with respect to a set $B \subset X$ provided that for each index $\alpha$ of $X$ there exists an index $\beta$ of $X$ such that if $y \in xT$ and $z \in B$ with $y \in z\beta$ then $yt \in x\alpha$ for all $t \in S$.

**Theorem 3.** Let $S$ be a replete semi-group in $T$. If the orbit of $x$ is $S$-Lyapunov stable with respect to $xT$ and $\text{Cl}(xS^{-1})$ is compact then $x$ is Bohr almost periodic.

**Proof.** If we can show under these hypotheses that $x$ is almost periodic under $T$ then by using Theorem 2 we can deduce that $x$ is Bohr almost periodic.

Let $S$ be a replete semi-group of $T$, let the orbit of $x$ be $S$-Lyapunov stable with respect to $xT$ and let $\text{Cl}(xS^{-1})$ be compact. It is clear that $S^{-1}$ is also a replete semi-group of $T$. Therefore, by the above remarks it follows that $S^{-1}_x$ is nonempty. Since $S^{-1}_x$ is closed it is compact. It follows from [6, 4.06] that there exists a $y \in S^{-1}_x$ such that $y$ is almost periodic. It follows from [6, 4.07] that $\text{Cl}(yT)$ is a compact minimal set. If $x \in \text{Cl}(yT)$ then $x$ is almost periodic and the theorem is proved.

Assume $x \in \text{Cl}(yT)$. Then there exists an index $\alpha$ of $X$ with the property that $x \in (\text{Cl}(yT))\alpha$. Let $\beta$ be an index of $X$ with the property that $\beta^2 \subset \alpha$. Since the orbit of $x$ is $S$-Lyapunov stable with respect to $xT$ there exists an index $\gamma$ of $X$ with the property that if $p \in q\gamma$ and $p, q \in xT$ then $pt \in q\beta$ for all $t \in S$. Let $\delta$ be a symmetric index of $X$ with the property that $\delta^2 \subset \gamma$. Since $y \in S^{-1}_x - xT$ and $S^{-1}_x = \bigcap_{t \in S} \text{Cl}(xtS^{-1})$ we have $y \in S^{-1}_x \subset \text{Cl}(xS^{-1})$. Hence there exists an $s_1 \in S$ such that $xs_1^{-1} \in y\delta$. Since $\pi$ is continuous there exists an index $\sigma$ of $X$ such that if $p \in yo$ then $ps \in y\beta$. There exists an $s \in S$ such that $xs_1^{-1} \in y\sigma \bigcap y\delta$. Thus $xs_1^{-1}s_1 \in y\beta$. Also $xs \in y\delta$ which implies that $xs_1^{-1} \in xs^{-1}\gamma$. Thus $xs_1^{-1}s_1 \in x_1^{-1}s_1\beta$. These two statements imply that
which is a contradiction. It follows that $x \in \text{Cl}(yT)$ and the theorem is proved.

A subset $E$ of $T$ is said to be $P$-extensive ($P$ is a replete semigroup not equal to $T$) [2, p. 1146] provided that $pP \cap E \neq \emptyset$ for all $p$ in $P$. A point $x \in X$ is said to be $P$-recurrent provided that for each index $\alpha$ of $X$ there exists a $P$-extensive set $E$ such that $xE \subset x\alpha$. Using these concepts and the previous theorem we are able to give a set of necessary and sufficient conditions in order for a point to be Bohr almost periodic.

**Theorem 4.** If $S_x$ is compact for some replete semigroup $S$ of $T$ then the following statements are equivalent:

1. $x$ is Bohr almost periodic,
2. $x$ is $S$-recurrent and the orbit of $x$ is $S$-Lyapunov stable with respect to $S_x$.

**Proof.** If $x$ is Bohr almost periodic and $\alpha$ is any index of $X$ then there exists a syndetic set $A \subset T$ such that $xtA \subset (xt)\alpha$ for all $t$ in $T$. It follows from Lemma 1 that $A$ is $P$-extensive for each replete semi-group of $T$. Hence $x$ is $S$-recurrent. From [1] it follows that $\text{Cl}(xT) = S_x$ and hence $\text{Cl}(xT)$ is compact. By [6, 4.37] it follows that the orbit of $x$ is $T$-Lyapunov stable with respect to $xT$. It follows in the same manner as in [7, p. 385] that the orbit of $x$ is $T$-Lyapunov stable with respect to $\text{Cl}(xT)$. Since $S_x = \text{Cl}(xT)$ the orbit of $x$ is $T$-Lyapunov stable with respect to $S_x$ which completes the proof of this half of the theorem.

If $x$ is $S$-recurrent then it follows that $\text{Cl}(xT) = S_x$. Hence $\text{Cl}(xT)$ is compact. Since the orbit of $x$ is $S$-Lyapunov stable with respect to $S_x$ it follows from Theorem 3 that $x$ is Bohr almost periodic.

An alternate proof of this theorem can be given using the main theorem in [3] and the theorem of Gottschalk [5] relating uniform almost periodicity and equi-continuity.

We now introduce the concept of $S$-orbital stability which is generalized from the notion of a final point being asymptotically stable which was discussed by Friedlander [4].

**Definition 6.** A point $y \in X$ is said to be $S$-orbitally stable ($SO$-stable) with respect to a set $B \subset X$ provided there exists an open set $U$ containing $y$ such that if $x \in B \cap U$ then $S_x = S_y$. When $X$ is a uniform space then the orbit of $y$ is uniformly $SO$-stable with respect to $B \subset X$ provided there exists an index $\alpha$ of $X$ such that if $x \in B \cap y\alpha$ then $S_x = S_y$. 

Lemma 2. If $S$ is a replete semi-group and $S_y$ is compact and nonempty then a necessary and sufficient condition that $S_y$ be a minimal set is that the orbit of $y$ be uniformly SO-stable with respect to $S_y$.

Proof. If $S_y$ is a minimal set then it follows immediately that the orbit of $y$ is uniformly SO-stable with respect to $S_y$.

Let the orbit of $y$ be uniformly SO-stable with respect to $S_y$ and $x \in S_y$. There exists an index $\delta$ of $X$ such that if $z \in S_y \cap y t \delta$ then $S_z = S_y$. Since $x \in S_y$ there exists a $t' \in T$ such that $x \in y t' \delta$ which implies $S_z = S_{y t'}$. But, since $S$ is a replete semi-group of $T$, $S_{y t'} = S_y$ hence $S_z = S_y$. Therefore, $S_z \subset \text{Cl}(xS) \subset \text{Cl}(xT)$ implies $S_y = \text{Cl}(xT)$ for all $x \in S_y$. Thus $S_y$ is a minimal set.

Theorem 5. Let $S$ be a replete semi-group in $T$ and let $S_y$ be compact and nonempty. If the orbit of $y$ is uniformly SO-stable and S-Lyapunov stable with respect to $S_y$ then $S_y$ is the closure of a Bohr almost periodic orbit.

Proof. It follows from the previous lemma that $S_y$ is a minimal set. By Theorem 4 it is sufficient to show that if $x \in S_y$ then the orbit of $x$ is S-Lyapunov stable with respect to $S_y = S_{x'}$. Let $\delta$ be an index of $X$ and $\beta$ be a symmetric index of $X$ such that $\beta^2 \subset \delta$. Since the orbit of $y$ is S-Lyapunov stable with respect to $S_y$ there exists an index $\gamma$ of $X$ such that if $z \in S_y \bigcap (yt') \gamma$ then $zt \in (yt') \beta$ for all $t \in S$. Let $\alpha$ be a symmetric index of $X$ with the property that $\alpha^2 \subset \gamma$. Let $z \in S_y \bigcap (xt') \alpha$. There exists a $\bar{t} \in T$ such that $y \bar{t} \in (xt') \alpha$ which implies $y \bar{t}t \in (xt') \beta$ and $zt \in (y \bar{t}t) \beta$ for all $t \in S$. Hence $zt \in (xt') \beta^2 \subset (xt') \delta$ for all $t \in S$. This implies that $x$ is S-Lyapunov stable with respect to $S_y$ and completes the proof of the theorem.

The question of necessary and sufficient conditions on $y$ in order that $S_y$ be the closure of a Bohr almost periodic point is still an open question. Theorem 5 shows that a necessary condition must be found on $y$ which will imply that $x \in S_y$ is uniformly S-Lyapunov stable with respect to $S_y$.

Bibliography


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