

# Pacific Journal of Mathematics

**A NOTE OF DILATIONS IN  $L^p$**

SAY ROMAN HARASYMIV

## A NOTE OF DILATIONS IN $L^p$

S. R. HARASYMIV

The objects of study in this note are the Lebesgue spaces  $L^p(1 < p < \infty)$  on the  $n$ -dimensional Euclidean space  $R^n$ . We consider a function  $f$  in one of the above-mentioned spaces, and derive results about the closure (in the relevant function space) of the set of linear combinations of functions of the form

$$f(a_1x_1 + b_1, \dots, a_nx_n + b_n)$$

where  $a_1, \dots, a_n, b_1, \dots, b_n \in R$ , and  $a_1 \neq 0, \dots, a_n \neq 0$ .

1. Notation and main results. The Haar measure on  $R^n$  will be denoted by  $dx$ . It will be assumed normalized so that the Fourier inversion formula holds without any multiplicative constants outside the integrals involved.

If  $x \in R^n$ , and  $k$  is an integer such that  $1 \leq k \leq n$ , then  $x_k$  will denote the  $k$ -th component of  $x$ . Multiplication (and of course addition) in  $R^n$  is defined component-wise, in the usual manner.

We write  $R^* = R^n \setminus \{x: x_k = 0 \text{ for some } k\}$ .

Suppose that  $1 < p < \infty$ . Then  $q$  will always be written for the number satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For each integer  $k$  such that  $1 \leq k \leq n$ ,  $J_k$  will denote the projection of  $R^n$  onto its  $k$ -th factor; *i.e.*

$$J_k(x) = x_k \text{ for all } x \in R^n.$$

If  $f$  is any function on  $R^n$ , and  $a \in R^*, b \in R^n$ , then  $f_b^a$  will denote the function defined by

$$f_b^a(x) = f(ax + b) \text{ for all } x \in R^n.$$

(The map  $x \rightarrow ax + b$  is called a dilation of  $R^n$ .) Finally, the set  $S_f$  is defined by

$$S_f = \{f_b^a: a \in R^*, b \in R^n\}.$$

In what follows, several vector spaces will be considered. If  $1 < p < \infty$ ,  $L^p(R^n)$  will denote the usual Lebesgue space.  $L^p(R^n)$  will be given the usual norm topology.

If  $f$  is an element of  $L^p(R^n)$  we shall denote by  $T[f]$  the closed vector subspace of  $L^p(R^n)$  generated by  $S_f$ .

Finally, if  $W$  is any open subset of  $R^n$ , we shall write  $C^\infty(W)$  for the space of functions defined on  $W$  and indefinitely differentiable there.  $\mathcal{D}(W)$  will denote the space of indefinitely differentiable functions with compact supports contained in  $W$ . The dual of the last space is the space  $\mathcal{D}'(W)$  of distributions on  $W$ . For details of these spaces see e.g. Schwartz [8].

Schwartz [7] considers the space of continuous functions on the  $n$ -dimensional Euclidean space  $R^n$  equipped with the topology of uniform convergence on compact sets. He shows that if  $f$  is a function in this space, and if the linear combinations of functions of the form

$$f(ax_1 + b_1, \dots, ax_n + b_n), \quad a, b_1, \dots, b_n \in R$$

are not dense in the space, then  $f$  satisfies at least one distributional equation of the form

$$P(D)f = 0$$

where  $P(D)$  is a nontrivial homogeneous linear partial differential operator with constant coefficients,

We shall prove the following result:

**THEOREM 1.** *If  $f \in L^p(R^n)$ , where  $1 < p < \infty$ , and  $f \neq 0$ , then  $\mathcal{T}[f] = L^p(R^n)$ .*

**2. Discussion of problem.** The Fourier transform  $\hat{g}$  of a function  $g$  in  $L^q(R^n)$  is defined as a distribution on  $R^n$ . (See, e.g., Schwartz [8]). It has the property of being locally a pseudomeasure; i.e., its restriction to a relatively compact open set  $W$  coincides with the restriction of some pseudomeasure to  $W$  (Gaudry [2] and [3]).

If  $W$  is an open set,  $g \in L^q(R)$ ,  $F \in \mathcal{D}'(W)$ , and if  $F$  coincides on each relatively compact open subset of  $W$  with the Fourier transform of an element of  $L^1(R^n)$ , then we define  $F \cdot \hat{g} \in \mathcal{D}'(W)$  by

$$F \cdot \hat{g}(\varphi) = \hat{g}(F\varphi) \text{ for all } \varphi \in \mathcal{D}(W).$$

It can be shown that if  $W$  is an open set,  $f \in L^p(R^n)$ ,  $g \in L^q(R^n)$ , and if  $\hat{f}$  coincides on each relatively compact open subset of  $W$  with the Fourier transform of an element in  $L^1(R^n)$ , then

$$\widehat{f * g} = \hat{f} \cdot \hat{g} \text{ on } W.$$

If  $f$  and  $h$  are in  $L^p(R^n)$ , then from the Hahn-Banach theorem it follows that  $h \in \mathcal{T}[f]$  if and only if

$$h * g(0) = 0$$

for all functions  $g$  in  $L^q(R^n)$  such that

$$(2.1) \quad f^\alpha * g = 0 \text{ for all } \alpha \in R^* .$$

Therefore, to establish Theorem 1, it is sufficient to prove the following assertion: if  $f \in L^p(R^n)$  for some  $p$  satisfying  $1 < p < \infty$ , and if  $g$  is such that (2.1) holds, then

$$(2.2) \quad \text{supp } \hat{g} \subseteq R^n \setminus R^* .$$

(We are bearing in mind the fact that  $R^n \setminus R^*$  is  $p$ -thin,  $1 < p \leq \infty$ . See Edwards [1].) The relation (2.2) will be established in § 4.

To prove (2.2), we shall show that if  $x \in R^*$ , then (2.1) implies the existence of a relatively compact neighbourhood  $W$  of  $x$ , and a function  $k \in L^1(R^n)$  such that

$$\hat{k} \cdot \hat{g} = 0$$

and  $|\hat{k}| > 0$  on  $\bar{W}$ . This will imply that  $\hat{g} = 0$  on  $W$ . For there will exist a function  $K \in L^1(R^n)$  such that

$$\hat{k} \hat{K} = 1 \text{ on } W$$

(Rudin [6], Theorem 2.6.2), and so if  $\varphi \in \mathcal{D}(W)$ , we have

$$\begin{aligned} \hat{g}(\varphi) &= \hat{g}(\hat{k} \hat{K} \varphi) \\ &= \hat{k} \hat{K} \cdot \hat{g}(\varphi) \\ &= \widehat{k * K * g}(\varphi) \\ &= 0 \end{aligned}$$

since  $k * g = 0$ . Section 3 is essentially devoted to constructing the required functions  $k$ .

**3. Preliminary results.** Consider any function  $\varphi \in \mathcal{D}(R^*)$ . Then if  $x \in R^*$ , it follows that  $\varphi^{x^{-1}} \in \mathcal{D}(R^*)$ . If  $s$  is any distribution on  $R^n$ , we define a function  $s \nabla \varphi$  on  $R^*$  by

$$s \nabla \varphi(x) = s(\varphi^{x^{-1}}) \text{ for all } x \in R^* .$$

We then have

**LEMMA 1.** *If  $\varphi \in \mathcal{D}(R^*)$  and  $s \in \mathcal{D}'(R^n)$ , then  $s \nabla \varphi \in C^\infty(R^*)$ .*

*Proof.* (cf. Hörmander [5], Theorem 1.6.1.)

First we show that  $s \nabla \varphi$  is continuous.

Suppose that  ${}^j x \rightarrow {}^0 x \in R^*$ . Then  $\varphi^{j x^{-1}} \rightarrow \varphi^{0 x^{-1}}$  in  $\mathcal{D}(R^*)$ . For let

$$\begin{aligned} a &= \sup \{ |x_k| : x \in \text{supp } \varphi, 1 \leq k \leq n \} < \infty \\ b &= \inf \{ |x_k| : x \in \text{supp } \varphi, 1 \leq k \leq n \} > 0 \end{aligned}$$

and let  $A, B > 0$  be numbers such that

$$B/b < |{}^j x_k| < A/a, \quad 1 \leq k \leq n$$

for all  $j$ . Then if  $y \in \text{supp } \varphi^{jx^{-1}}$ , we have  $y/{}^j x \in \text{supp } \varphi$ . This implies that

$$b \leq |y_k/{}^j x_k| \leq a, \quad 1 \leq k \leq n,$$

and so

$$|{}^j x_k| b \leq |y_k| \leq |{}^j x_k| a, \quad 1 \leq k \leq n.$$

It follows that  $B < |y_k| < A$ . Hence all the sets  $\text{supp } \varphi^{jx^{-1}}$  are contained in a fixed compact subset of  $R^*$ . Furthermore, since

$$D_k(\varphi^{x^{-1}}) = \frac{1}{x_k} (D_k \varphi)^{x^{-1}}, \quad 1 \leq k \leq n,$$

it is easily shown that for each multi-index  $\alpha$ ,

$$\lim_j D^\alpha(\varphi^{jx^{-1}}) = D^\alpha(\varphi^{0x^{-1}})$$

uniformly. Thus  $\varphi^{jx^{-1}} \rightarrow \varphi^{0x^{-1}}$  in  $D(R^*)$  and, since  $s$  is continuous, we have

$$\lim_j s \nabla \varphi({}^j x) = s \nabla \varphi({}^0 x).$$

Hence  $s \nabla \varphi$  is continuous on  $R^*$ .

To complete the proof of the lemma, it is sufficient, in view of the above, to show that if  $1 \leq k \leq n$ , then

$$(3.1) \quad D_k(s \nabla \varphi) = -1/J_k \cdot s \nabla J_k D_k \varphi \text{ on } R^* .$$

The required result will then follow by induction.

Thus, let  $e_k$  be the unit vector along the  $k$ -axis and consider the quotient

$$[s \nabla \varphi(x + h e_k) - s \nabla \varphi(x)]/h = s[\varphi^{(x+h e_k)^{-1}} - \varphi^{x^{-1}}]/h$$

where  $x \in R^*$  and  $h \neq 0$ . We have

$$(3.2) \quad \lim_{h \rightarrow 0} [\varphi^{(x+h e_k)^{-1}} - \varphi^{x^{-1}}]/h = -1/x_k \cdot (J_k D_k \varphi)^{x^{-1}} \text{ in } D(R^*) .$$

To verify this, consider any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$ . We have

$$\begin{aligned} & D^\alpha[\varphi^{(x+h e_k)^{-1}} - \varphi^{x^{-1}}]/h \\ &= \left[ 1/\prod_{j \neq k} x_j^{\alpha_j} \right] [(1/(x_k + h)^{\alpha_k}) \cdot (D^\alpha \varphi)^{(x+h e_k)^{-1}} - (1/x_k^{\alpha_k}) \cdot (D^\alpha \varphi)^{x^{-1}}]/h . \end{aligned}$$

The last expression converges pointwise to  $D^\alpha[-(1/x_k)(J_k D_k \varphi)^{x^{-1}}]$ .

The convergence is in fact uniform. This may be deduced from the fact that if  $\psi$  is any function in  $\mathbf{D}(R^n)$ , and  $h$  is a positive number, then

$$|[\psi(y + he_k) - \psi(y)]/h - D_k\psi(y)| < |h| \cdot \|D_k^2\psi\|_\infty$$

which follows easily via the mean-value theorem. This establishes (3.2), and (3.1) follows. Thus the proof of Lemma 1 is complete.

**COROLLARY.** *If  $W$  is any relatively compact open set such that  $\bar{W} \subseteq R^*$ , and if  $\varphi \in \mathbf{D}(R^*)$  and  $s \in \mathbf{D}'(R^n)$ , then there exists a function  $k$  in  $L^1(R^n)$  such that*

$$s \nabla \psi = \hat{k} \text{ on } W .$$

*Proof.* In fact we may take for  $k$  any function of the form

$$(s \nabla \varphi \cdot \psi)^\vee$$

where  $\psi \in \mathbf{D}(R^*)$  and  $\psi = 1$  on  $\bar{W}$ . [Here and elsewhere,  $\vee$  denotes the inverse Fourier transform:

$$\check{h}(x) = \int_{R^n} e^{2\pi ixy} h(y) dy \text{ for all } h \in L^1(R^n) .$$

**LEMMA 2.** *If  $f \in L^p(R^n)$ ,  $g \in L^q(R^n)$  and  $\varphi, \psi \in \mathbf{D}(R^*)$ , then*

$$\hat{f} \nabla \varphi \cdot \hat{g}(\psi) = \int_{R^n} (\varphi(t)/|J_1(t)| \cdots |J_n(t)|) \left\{ \int_{R^n} \widehat{\psi |J_1 \cdots J_n|}(x) \cdot g * f^{t-1}(x) dx \right\} dt .$$

*Proof.* Choose a sequence  $\{f_j\}$  of functions in  $L^1(R^n) \cap L^p(R^n)$  such that

$$\lim_j f_j = f \text{ in } L^p(R^n) .$$

Then, if  $\psi$  is any function in  $\mathbf{D}(R^*)$ , we have

$$(3.3) \quad \lim_j \hat{f}_j \nabla \varphi \cdot \psi = \hat{f} \nabla \varphi \cdot \psi \text{ in } \mathbf{D}(R^*) .$$

For, if  $\alpha$  is any multi-index, the Leibnitz formula for the differentiation of a product shows that  $D^\alpha[(\hat{f}_j - \hat{f}) \nabla \varphi \cdot \psi]$  is a sum of terms of the form

$$A \cdot D^\beta[(\hat{f}_j - \hat{f}) \nabla \varphi] D^{\alpha-\beta} \psi$$

where  $\beta_i \leq \alpha_i, i = 1, \dots, n$ , and  $A$  is a constant depending only on  $\alpha$  and  $\beta$ . Thus we are reduced to proving that if  $\alpha$  is any multi-index, then

$$(3.4) \quad \lim_j D^\alpha[(\hat{f}_j - \hat{f}) \nabla \varphi] = 0 \text{ uniformly on } \text{supp } \psi .$$

Now, quite generally, if  $s$  is a distribution on  $R^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  a multi-index, and  $\varphi$  a function in  $\mathbf{D}(R^n)$ , then

$$D^\alpha(s \nabla \varphi) = (1/J_1^{\alpha_1} \dots J_n^{\alpha_n}) \sum_{\beta} \{a_{\beta} s \nabla (J_1^{\beta_1} \dots J_n^{\beta_n} D^{\beta} \varphi)\}$$

where the  $a_{\beta}$  are constants depending only on  $\alpha$  and  $\beta$ , and the summation is carried out over all multi-indices  $\beta$  such that  $\beta_1 \leq \alpha_1, \dots, \beta_n \leq \alpha_n$ . This is easily shown by induction, using (3.1). Since  $J_1, \dots, J_n$  are bounded away from zero on  $\text{supp } \psi$ , it suffices, in order to establish (3.4), to show that for every multi-index  $\alpha$

$$(3.5) \quad \lim_j (\hat{f}_j - \hat{f}) \nabla (J_1^{\alpha_1} \dots J_n^{\alpha_n} D^{\alpha} \varphi) = 0 \text{ uniformly on } \text{supp } \psi .$$

Thus, let

$$a = \sup \{ |x_k| : x \in \text{supp } \psi, 1 \leq k \leq n \} .$$

Then, if  $x \in \text{supp } \psi$ , we have

$$\begin{aligned} & |[(\hat{f}_j - \hat{f}) \nabla (J_1^{\alpha_1} \dots J_n^{\alpha_n} D^{\alpha} \varphi)](x)| \\ &= \left| \int_{R^n} (f_j - f)(yx^{-1}) \cdot \widehat{J_1^{\alpha_1} \dots J_n^{\alpha_n} D^{\alpha} \varphi}(y) dy \right| \\ &\leq \|f_j - f\|_p \cdot a^n \cdot \|\widehat{J_1^{\alpha_1} \dots J_n^{\alpha_n} D^{\alpha} \varphi}\|_q \end{aligned}$$

From this, (3.5) follows immediately, and hence (3.4) and (3.3).

Using (3.3), it is seen that

$$(3.6) \quad \begin{aligned} \hat{f} \nabla \varphi \cdot \hat{g}(\psi) &= \lim_j \hat{g}(\hat{f}_j \nabla \varphi \cdot \psi) \\ &= \lim_j \int_{R^n} g(x) \cdot [\hat{f}_j \nabla \varphi \cdot \psi]^{\vee}(-x) dx . \end{aligned}$$

Now

$$\begin{aligned} & [\hat{f}_j \nabla \varphi \cdot \psi]^{\vee}(-x) \\ &= \int_{R^n} e^{-2\pi ixy} \hat{f}_j \nabla \varphi(y) \cdot \psi(y) dy \\ &= \int_{R^n} e^{-2\pi ixy} \left\{ \int_{R^n} \hat{f}_j(t) \varphi(ty^{-1}) dt \right\} \psi(y) dy \\ &= \int_{R^n} e^{-2\pi ixy} \psi(y) \left\{ \int_{R^n} \hat{f}_j(yt) \varphi(t) |J_1(y)| \dots |J_n(y)| dt \right\} dy \\ &= \int_{R^n} \varphi(t) \left\{ \int_{R^n} \hat{f}_j(yt) \psi(y) |J_1(y)| \dots |J_n(y)| e^{-2\pi ixy} dy \right\} dt \\ &= \int_{R^n} (\varphi(t) / |J_1(t)| \dots |J_n(t)|) \cdot \widehat{\psi} |J_1 \dots J_n| * f_j^{t-1}(x) dt . \end{aligned}$$

Substituting this in (3.6), we have

$$\begin{aligned}
 & f \nabla \varphi \cdot \hat{g}(\psi) \\
 (3.7) \quad &= \lim_j \int_{R^n} g(x) \left\{ \int_{R^n} (\varphi(t) / |J_1(t)| \cdots |J_n(t)|) \widehat{\psi | J_1 \cdots J_n} | * f_j^{t^{-1}}(x) dt \right\} dx \\
 &= \lim_j \int_{R^n} (\varphi(t) / |J_1(t)| \cdots |J_n(t)|) \left\{ \int_{R^n} \widehat{\psi | J_1 \cdots J_n} | (x) g * f_j^{t^{-1}}(x) dx \right\} dt.
 \end{aligned}$$

Now, if

$$a = \sup \{ |t_k| : t \in \text{supp } \varphi, 1 \leq k \leq n \},$$

then if  $t \in \text{supp } \varphi$ , we have

$$\begin{aligned}
 & \left| \int_{R^n} \widehat{\psi | J_1 \cdots J_n} | (x) \cdot g * f_j^{t^{-1}}(x) dx - \int_{R^n} \widehat{\psi | J_1 \cdots J_n} | (x) g * f^{t^{-1}} dx \right| \\
 & \leq \| \widehat{\psi | J_1 \cdots J_n} \|_1 \cdot \| g * (f_j^{t^{-1}} - f^{t^{-1}}) \|_\infty \\
 & \leq \| \widehat{\psi | J_1 \cdots J_n} \|_1 \cdot \| g \|_q \cdot \| f_j^{t^{-1}} - f^{t^{-1}} \|_p \\
 & \leq \| \widehat{\psi | J_1 \cdots J_n} \|_1 \cdot \| g \|_q \| f_j - f \|_p \cdot a^n.
 \end{aligned}$$

Using this, and (3.7) we see that

$$\begin{aligned}
 & \hat{f} \nabla \varphi \cdot \hat{g}(\psi) \\
 &= \int_{R^n} (\varphi(t) / |J_1(t)| \cdots |J_n(t)|) \left\{ \int_{R^n} \widehat{\psi | J_1 \cdots J_n} | (x) \cdot g * f^{t^{-1}}(x) dx \right\} dt
 \end{aligned}$$

This completes the proof of Lemma 2.

**COROLLARY.** *If  $\varphi \in D(R^*)$ ,  $f \in L^p(R^n)$ ,  $g \in L^q(R^n)$ , and if*

$$f^a * g = 0 \text{ for all } a \in R^*$$

*then  $\hat{f} \nabla \varphi \cdot \hat{g} = 0$  on  $R^*$ .*

**LEMMA 3.** *Suppose that  $f \in L^p(R^n)$  and that  $R^* \cap \text{supp } \hat{f} \neq \emptyset$ . Then if  $g \in L^q(R^n)$  is such that*

$$f^a * g = 0 \text{ for all } a \in R^*,$$

*We have*

$$\text{supp } \hat{g} \subseteq R^n \setminus R^*.$$

*Proof.* First we observe that

$$\text{supp } \hat{f}^a = a \cdot \text{supp } \hat{f}$$

and hence, since  $R^* \cap \text{supp } \hat{f} \neq \emptyset$ ,



$$(3.8) \quad \bigcup_{a \in R^*} \text{supp } \widehat{f^a} \subseteq R^* .$$

Now suppose that  $x \in R^*$ . By (3.8),  $x \in \text{supp } \widehat{f^b}$  (say). Choose a relatively compact neighbourhood  $W$  of  $x$  such that  $\overline{W} \subseteq R^*$ . There exists a function  $\varphi \in \mathcal{D}(W)$  such that

$$\widehat{f^b}(\varphi) \neq 0$$

i.e. 
$$\widehat{f^b} \nabla \varphi^{x^{-1}}(x) \neq 0 .$$

This implies that  $\widehat{f^b} \nabla \varphi^{x^{-1}}$  is bounded away from 0 on a neighbourhood of  $x$ . Since  $\widehat{f^b} \nabla \varphi^{x^{-1}} \in C^\infty(R^*)$  (by Lemma 1) and (by the corollary to Lemma 2)

$$\widehat{f^b} \nabla \varphi^{x^{-1}} \cdot \widehat{g} = 0 \text{ on } R^*$$

the corollary to Lemma 1 and the reasoning indicated in § 2 together entail that  $x \notin \text{supp } \widehat{g}$ . Thus

$$\text{supp } \widehat{g} \subseteq R^n \setminus R^*$$

as we wished to show.

**4. Proof of Theorem 1.** We can now prove Theorem 1.

Let  $f \in L^p(R^n)$ , ( $1 < p < \infty$ ),  $f \neq 0$ , and suppose that  $g \in L^q(R^n)$  is such that

$$f^a * g = 0 \text{ for all } a \in R^* .$$

Since  $R^n \setminus R^*$  is  $q$ -thin if  $1 < q < \infty$ , we deduce that

$$\text{supp } \widehat{f} \cap R^* \neq \emptyset .$$

Then, by Lemma 3,

$$\text{supp } \widehat{g} \subseteq R^n \setminus R^*$$

and so (since  $R^n \setminus R^*$  is  $p$ -thin)  $g = 0$ .

I take this opportunity to thank my supervisor, Dr. R. E. Edwards, in the first place for suggesting this problem, and secondly, for his suggestions and criticisms concerning the work presented here.

REFERENCES

1. R. E. Edwards, *Spans of translates in  $L^p$* , J. Aust. Math. Soc. V (1965), 216-233.
2. G. Gaudry, *Quasimeasures and operators commuting with convolution* (to appear, Pacific J. Math.)
3. ———, *Multipliers of type  $(p, q)$*  (to appear, Pacific J. Math.)

4. C. S. Herz, *Spectral theory of bounded functions*, Trans. Amer. Math. Soc. **94** (1960), 181-232.
5. L. Hormander, *Linear Partial Differential Operators*, Academic Press, New York, 1963.
6. W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, New York, 1962.
7. Laurent Schwartz, *Sur Certaines Familles Non Fondamentales de Fonctions Continues*, Bull. Soc. Math. France **72** (1944), 141-145.
8. ———, *Théorie des Distributions*, Hermann, Paris.

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