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ITERATES OF BERNSTEIN POLYNOMIALS

RICHARD PAUL KELISKY AND THEODORE JOSEPH RIVLIN

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R. P. KELISKY AND T. J. RIVLIN

$B_n(f)$ transforms each function defined on $[0, 1]$ into its Bernstein polynomial of degree n . In this paper we study the convergence of the iterates $B_n^{(k)}(f)$ as $k \rightarrow \infty$ both in the case that k is independent of n and (for polynomial f) when k is a function of n .

To each $f(x)$ defined on $I: 0 \leq x \leq 1$ there is associated its Bernstein polynomial of degree n defined by

$$(1.1) \quad B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

It is well known that if f is continuous on I , then

$$(1.2) \quad \lim_{n \rightarrow \infty} B_n(f; x) = f(x)$$

uniformly on I . (Cf., Lorentz [2] for this and other properties of the Bernstein polynomials used here.) Let $B_n(f)$ denote the (polynomial) function defined by (1.1), then for $k > 1$, $B_n^{(k)}(f; x) = B_n(B_n^{(k-1)}(f); x)$ defines, by mathematical induction, a sequence of iterates of the Bernstein polynomials. Our purpose is to study the convergence behavior of this sequence as $k \rightarrow \infty$, both in the case that k is independent of n and when it is a nonconstant function of n .

We show in § 2 that $B_n^{(k)}(f; x)$ converges (uniformly) for fixed n , to the line segment joining $(0, f(0))$ to $(1, f(1))$, and in § 3 that the sequence $B_n^{(g(n))}(x^s; x)$ with appropriate assumptions on $g(n)$, also converges, for each $s = 0, 1, 2, \dots$ to a polynomial of degree s whose coefficients we determine explicitly. Finally, in § 4 arbitrary iterates are defined as a natural generalization of the positive integral iterates.

When (1.1) is rewritten in conventional polynomial form, it becomes

$$(1.3) \quad \begin{aligned} B_n(f; x) &= \sum_{q=0}^n \left\{ \binom{n}{q} \sum_{k=0}^q f\left(\frac{k}{n}\right) \binom{q}{k} (-1)^{q-k} \right\} x^q \\ &= \sum_{q=0}^n \Delta_{1/n}^q f(0) \binom{n}{q} x^q \end{aligned}$$

which reveals that if f is a polynomial of degree m , then $B_n(f)$ is a polynomial whose degree is at most $\min(m, n)$. Let s be a fixed positive integer satisfying $s \leq n$. (There is no loss of generality in this restriction on s for $k > 1$, since for $s > n$, $B_n^{(k)}(x^s) = B_n^{(k-1)}(B_n(x^s))$ and $B_n(x^s)$ is of degree at most n .) We consider $f(x) = x^j, j = 1, \dots, s$. (1.3) implies that

$$(1.4) \quad B_n(x^j) = a_{1j}x + a_{2j}x^2 + \cdots + a_{jj}x^j = \sum_{q=1}^j \pi_q \sigma_j^q \frac{1}{n^{j-q}} x^q, \\ j = 1, \dots, s,$$

where σ_j^q are the Stirling numbers of the second kind (Cf., Jordan [1, pp. 168-173]) defined by

$$(1.5) \quad \sigma_j^q = \frac{(-1)^q}{q!} \sum_{k=1}^q k^j \binom{q}{k} (-1)^k,$$

and

$$(1.6) \quad \begin{cases} \pi_q = \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{q-1}{n}\right), & q = 2, \dots, s \\ \pi_1 = 1. \end{cases}$$

2. Limit of the iterates. The study of the iterates of $B_n(f; x)$ for $f(x) = x^s$ is considerably simplified if we use the language of linear algebra. There is no loss of generality in this choice of $f(x)$ since B_n replaces f by a polynomial.

Let A denote the $s \times s$ upper triangular matrix whose entries a_{ij} are defined in (1.4), i.e.,

$$(2.1) \quad a_{ij} = \begin{cases} \pi_j \sigma_j^i n^{i-j}, & i \leq j \\ 0 & i > j. \end{cases}$$

Let e_s be the column vector of s components, the first $s - 1$ components being zero and the last one. Then

LEMMA 1. *If $A^k e_s = (\alpha_1^{(k)}, \dots, \alpha_s^{(k)})^T$, then*

$$(2.2) \quad B_n^{(k)}(x^s) = \alpha_1^{(k)}x + \alpha_2^{(k)}x^2 + \cdots + \alpha_s^{(k)}x^s, \quad k = 1, 2, \dots.$$

Proof. If $p(x) = c_1x + c_2x^2 + \cdots + c_sx^s$ (for example, $p(x) = B_n^{(j)}(x^s)$) and

$$B_n(p) = d_1x + d_2x^2 + \cdots + d_sx^s = \sum_{j=1}^s c_j (a_{1j}x + \cdots + a_{sj}x^s) \\ = \sum_{l=1}^s \sum_{j=1}^s c_j a_{lj} x^l,$$

then $(d_1, \dots, d_s)^T = A(c_1, \dots, c_s)^T$. The lemma now follows by mathematical induction on k .

LEMMA 2. *The eigenvalues of A are $\pi_1, \pi_2, \dots, \pi_s$.*

Proof. $a_{ii} = \pi_i, i = 1, \dots, s$, and $a_{ij} = 0$ if $i > j$.

Let A denote the $s \times s$ matrix with the eigenvalues of A , π_1, \dots, π_s , on the main diagonal and zeros everywhere else. Let V denote the matrix of eigenvectors of A , normalized so that the entries on its main diagonal are all 1. V is upper triangular and its entries are, in general, functions of n . Since $AV = VA$ we conclude that

$$(2.3) \quad A^k = VA^kV^{-1}.$$

Essentially, the following arguments rest on the observation that A^k is known to us and V and its inverse are independent of k .

LEMMA 3. *If $V^{-1} = (\bar{v}_{ij})$ then $\bar{v}_{ij} = 1, j = 1, \dots, s$.*

Proof. Let U be the eigenmatrix of A^T , i.e.,

$$A^T U = UA.$$

Let U (which is lower triangular) be normalized so that the entries on its main diagonal are all 1. Since $B_n(x^j; 1) = 1$ the column sums of A are all 1 and hence the row sums of A^T are all 1. The first column of U is the eigenvector associated with the eigenvalue $\pi_1 = 1$, and hence consists of all entries 1. Due to the way we have normalized V and U we know that $U^T = V^{-1}$ and the lemma is proved.

LEMMA 4. *If n is fixed*

$$\lim_{k \rightarrow \infty} A^k e_s = (1, 0, 0, \dots, 0)^T.$$

Proof. The entries on the main diagonal of A^k are π_1^k, \dots, π_s^k and

$$\begin{aligned} \lim_{k \rightarrow \infty} \pi_j^k &= 0, & j &= 2, \dots, s \\ \lim_{k \rightarrow \infty} \pi_1^k &= 1. \end{aligned}$$

Thus, as $k \rightarrow \infty$, VA^kV^{-1} approaches a matrix whose first row consists of all 1's, by Lemma 3, and the rest of whose elements are all 0. Clearly,

$$(1, 0, 0, \dots, 0)^T = \left(\lim_{k \rightarrow \infty} A^k \right) e_s = \lim_{k \rightarrow \infty} (A^k e_s).$$

THEOREM 1. *If n is fixed then*

$$(2.4) \quad \lim_{j \rightarrow \infty} B_n^{(j)}(f; x) = f(0) + (f(1) - f(0))x, \quad 0 \leq x \leq 1.$$

Proof. Let $B_n(f; x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$, then

$$B_n^{(j)}(f; x) = \alpha_0 + \alpha_1 B_n^{(j-1)}(x; x) + \alpha_2 B_n^{(j-1)}(x^2; x) + \dots + \alpha_n B_n^{(j-1)}(x^n; x);$$

hence, in view of Lemma 1 and Lemma 4, with $s = 1, 2, \dots, n$,

$$\begin{aligned}\lim_{j \rightarrow \infty} B_n^{(j)}(f; x) &= \alpha_0 + (\alpha_1 + \dots + \alpha_n)x \\ &= f(0) + (f(1) - f(0))x.\end{aligned}$$

REMARK. The convergence in (2.4) is uniform since we have a sequence of polynomials of fixed degree approaching a fixed polynomial of the same degree for all x on a bounded interval. Also we have used the obvious fact that $B_n(1) = 1$, all n .

It is a curious fact that the matrix V has the property that v_{ij} is independent of n , for $j = 1, 2, 3$. We have, when $s = 3$,

$$V = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $p_2(x) = -x + x^2$ and $p_3(x) = (1/2)x - (3/2)x^2 + x^3$, then we conclude that,

$$B_n^{(j)}(p_2) = \left(1 - \frac{1}{n}\right)^j p_2, \quad j = 0, 1, 2, \dots$$

$$B_n^{(j)}(p_3) = \left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\right]^j p_3.$$

These results should be contrasted to the well-known remark (Cf., Schoenberg [3]) that the Bernstein operators are "poor reproducers", in that they never reproduce polynomials of degree greater than 1.

3. **Limit of the coupled iterates.** Suppose $f(x) = x^s$. Theorem 1 tells us that for fixed n , $B_n^{(j)}(x^s) \rightarrow x$ as $j \rightarrow \infty$, while according to (1.2), $B_n(x^s) \rightarrow x^s$ as $n \rightarrow \infty$. Thus, it is of interest to "play-off" the upper and lower subscripts in $B_n^{(j)}(x^s)$, by considering $j = g(n)$. To this end we must examine the behavior of the eigenmatrix, V , as $n \rightarrow \infty$.

Let the elements of V be $v_{ij}(=v_{ij}(n))$. For $j = 1, \dots, s$ we have

$$(3.1) \quad A(v_{1j}, \dots, v_{sj})^T = \pi_j(v_{1j}, \dots, v_{sj})^T.$$

We examine these linear equations more closely. Since V is upper triangular,

$$(3.2) \quad v_{ij} = 0, \quad i = j + 1, \dots, s,$$

and because of the way we have normalized V

$$(3.3) \quad v_{jj} = 1.$$

It remains, then, to determine the behavior of $v_{ij}(n)$, $i < j$, as $n \rightarrow \infty$.

We consider the relevant linear equations from (3.1) (and write v_i in place of v_{ij} for simplicity)

$$(3.4) \quad \begin{aligned} & a_{j-1,j-1}v_{j-1} + a_{j-1,j} = \pi_j v_{j-1} \\ & a_{j-2,j-2}v_{j-2} + a_{j-2,j-1}v_{j-1} + a_{j-2,j} = \pi_j v_{j-2} \\ & \quad \vdots \\ & a_{11}v_1 + a_{12}v_2 + \cdots + a_{1,j-1}v_{j-1} + a_{1,j} = \pi_j v_1. \end{aligned}$$

Define $\pi_{ij} = \pi_i - \pi_j$, let P denote the determinant $|p_{ij}|$ such that

$$p_{ij} = \begin{cases} a_{ij} & i < j \\ \pi_{ij} & i = j \\ 0 & i > j \end{cases},$$

then

$$P = \prod_{k=1}^{j-1} \pi_{kj}.$$

Let $P^{(i)}$ denote the determinant identical to P except that the i -th column of P is replaced by $(-a_{1j}, -a_{2j}, \dots, -a_{j-1,j})$. Then, if we solve (3.4) for $v_i (= v_{i,j})$ by Cramer's rule, we obtain

$$(3.5) \quad v_i = \frac{P^{(i)}}{P}.$$

If we denote by $P_{pj}^{(i)}$ the minor of $-a_{pj}$ in $P^{(i)}$, then $P_{pj}^{(i)}$ is upper triangular and

$$(-1)^{i+p} P_{pj}^{(i)} = \begin{cases} 0 & p < i \\ P/\pi_{ij} & p = i \\ a_{i,i+1}a_{i+1,i+2} \cdots a_{p-1,p} P / \prod_{k=i}^p \pi_{kj} & p > i. \end{cases}$$

Now,

$$(3.6) \quad (-1)^{i+p+1} a_{pj} P_{pj}^{(i)} / P = \begin{cases} -a_{ij}/\pi_{ij} & p = i \\ \frac{(-1)^{i+p+1} a_{pj} a_{i,i+1} \cdots a_{p-1,p}}{\prod_{k=i}^p \pi_{kj}} & p > i, \end{cases}$$

and for $q < j$,

$$(3.7) \quad \begin{aligned} \pi_{qj} &= \pi_q \left[1 - (1 - q/n) \cdots \left(1 - \frac{j-1}{n} \right) \right] \\ &= \pi_q \left\{ \frac{1}{n} [q + (q+1) + \cdots + (j-1)] + O(n^{-2}) \right\} \end{aligned}$$

as $n \rightarrow \infty$. Since $\pi_i \rightarrow 1$ as $n \rightarrow \infty$, we obtain, in view of (3.6), (3.7), and (2.1),

$$\lim_{n \rightarrow \infty} \frac{\alpha_{pj} P_{pj}^{(i)}}{P} = 0, \quad p < j - 1,$$

while

$$\lim_{n \rightarrow \infty} \frac{\alpha_{j-1,j} P_{j-1,j}^{(i)}}{P} = \left\{ \prod_{t=i}^{j-1} \left(\frac{j-t}{2} \right) (j+t-1) \right\}^{-1} \sigma_{t+1}^t.$$

Thus, we obtain, finally, that

$$(3.8) \quad \lim_{n \rightarrow \infty} v_{ij} = v_{ij}^* = (-1)^{j+i} 2^{j-i} \frac{\prod_{t=i}^{j-1} \binom{t+1}{2}}{[(j-i)!]^2 \binom{2j-2}{j-1}}, \quad i = 1, \dots, j-1.$$

where we have used the fact that (Cf., Jordan [1])

$$\sigma_{t+1}^t = \binom{t+1}{2}.$$

(3.2), (3.3), and (3.8) give the limit of V as $n \rightarrow \infty$. In an entirely analogous fashion, with A^x in place of A , we may obtain the limit of V^{-1} as $n \rightarrow \infty$. We suppress the details, but the result is

$$(3.9) \quad \lim_{n \rightarrow \infty} \bar{v}_{ij} = \bar{v}_{ij}^* = \begin{cases} 0, & i > j \\ 1, & i = j \\ 2^{j-i} \frac{\prod_{t=i}^{j-1} \binom{t+1}{2}}{[(j-i)!]^2 \binom{i+j-1}{j-i}}, & i < j. \end{cases}$$

Let us put

$$(3.10) \quad E_j = \exp \left[-\binom{j}{2} \right] = \lim_{n \rightarrow \infty} \pi_j^n.$$

THEOREM 2. Suppose $g(n)$ is a nonnegative integer for each n , and

$$(3.11) \quad \lim_{n \rightarrow \infty} \frac{g(n)}{n} = \alpha,$$

then we have

$$(3.12) \quad \lim_{n \rightarrow \infty} B_n^{(g(n))}(x^s) = \sum_{i=1}^s b_i x^i$$

where

$$(3.13) \quad b_i = \frac{i}{s} \binom{s}{i}^2 \sum_{j=i}^s \frac{(-1)^{j+i} \binom{s-i}{j-i}^2}{\binom{2j-2}{j-i} \binom{j+s-1}{s-j}} E_j^\alpha,$$

$i = 1, \dots, s$ (where, when $\alpha = \infty$ in (3.11), we have $E_1^\alpha = 1$ and $E_j^\alpha = 0, j > 1$ in (3.13)).

Proof. $A^{g(n)} = V A^{g(n)} V^{-1}$. Now

$$\lim_{n \rightarrow \infty} A^{g(n)} = A^*$$

where A^* is a diagonal matrix with entries $E_j^\alpha, j = 1, \dots, s$ on its main diagonal.

Let

$$\lim_{n \rightarrow \infty} V = V^*$$

and

$$\lim_{n \rightarrow \infty} V^{-1} = (V^{-1})^* = (V^*)^{-1}.$$

The entries in V^* and $(V^*)^{-1}$ are given by (3.2), (3.3), (3.8), and (3.9). Thus, we may conclude that

$$V^* A^* (V^*)^{-1} e_s = \left(\lim_{n \rightarrow \infty} A^{g(n)} \right) e_s = \lim_{n \rightarrow \infty} (A^{g(n)} e_s)$$

and the existence of the limit in (3.12) is established. In order to verify (3.13), we need only note that

$$(3.14) \quad (b_1, \dots, b_s)^T = V^* A^* (V^*)^{-1} e_s,$$

so that

$$(3.15) \quad b_i = \sum_{j=1}^s v_{i,j}^* \bar{v}_{j,s}^* E_j^\alpha, \quad i = 1, \dots, s.$$

REMARK. If $\alpha = 0$, then $A^* = I$ and we conclude from (3.14) that $(b_1, \dots, b_s)^T = e_s$, or $b_j = 0, j = 1, \dots, s-1, b_s = 1$. In particular, then, if $g(n) \equiv 0$, we have proved (1.2) for the case $f(x) = x^s$. As a curiosity we also note that we have established the seemingly nontrivial identities

$$(3.16) \quad \sum_{j=i}^s \frac{(-1)^{j+i} \binom{s-i}{j-i}^2}{\binom{2j-2}{j-i} \binom{j+s-1}{s-j}} = 0, \quad i = 1, \dots, s-1.$$

With some simplification (3.16) may be written in the equivalent form (3.17) which holds for odd t and n positive

$$(3.17) \quad \sum_{k=0}^n (-1)^k \binom{t+k}{t} \binom{2n+t}{n-k} \frac{2k+t}{k+t} = 0.$$

Additionally, since

$$\sum_{i=1}^s a_{ij} = 1, \quad j = 1, \dots, s$$

and

$$\sum_{j=1}^s a_{ij} v_{jk} = \pi_k v_{ik}, \quad i = 1, \dots, s; \quad k = 1, \dots, s,$$

we obtain, after summing on i on both sides of (3.18) and interchanging the order of summation on the left

$$\sum_{j=1}^s v_{jk} = \pi_k \sum_{i=1}^s v_{ik},$$

from which we conclude that, if δ_{jk} is a Kronecker delta.

$$\sum_{i=1}^s v_{ik} = \delta_{1k}$$

and hence also

$$\sum_{i=1}^s v_{ik}^* = \delta_{1k}.$$

We thus have the seemingly nontrivial identities:

$$(3.19) \quad 1 + \sum_{i=1}^{j-1} (-1)^{j+i} 2^{j-i} \frac{\prod_{t=i}^{j-1} \binom{t+1}{2}}{[(j-i)!]^2 \binom{2j-2}{j-i}} = 0, \quad j = 2, \dots, s,$$

or, equivalently, if $n \geq 1$,

$$(3.20) \quad \sum_{k=0}^n (-1)^k \binom{n+k}{k} \binom{n}{k} \frac{1}{k+1} = 0.$$

4. Iterates of all orders. If t is any real number, $-\infty < t < \infty$, we are now in a position to define $B_n^{(t)}(f)$, in a manner consistent

with our definition when t is a nonnegative integer. We define

$$(4.1) \quad B_n^{(t)}(x^k) = b_1(t)x + b_2(t)x^2 + \cdots + b_k(t)x^k, \quad k = 1, 2, \dots,$$

where

$$(4.2) \quad (b_1(t), \dots, b_k(t))^T = V A^t V^{-1} e_k.$$

In (4.2), A^t is defined to be the diagonal $k \times k$ matrix whose entries on the main diagonal are $\pi_1^t, \pi_2^t, \dots, \pi_k^t$. It now follows that, since e_1, \dots, e_s is a basis in E^s ($s \leq n$), if

$$(4.3) \quad p = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_s x^s,$$

then

$$(4.4) \quad B_n^{(t)}(p) = \sum_{i=1}^s \alpha_i B_n^{(t)}(x^i).$$

Moreover, if we define

$$(4.5) \quad B_n^{(t)}(c) = c$$

and

$$(4.6) \quad B_n^{(t)}(c + p) = c + B_n^{(t)}(p)$$

where c is a constant and p is given by (4.3), then we obtain

$$(4.7) \quad B_n^{(t)}(p) = \sum_{i=0}^s \alpha_i B_n^{(t)}(x^i)$$

when

$$p = \alpha_0 + \alpha_1 x + \cdots + \alpha_s x^s.$$

We observe further that if $-\infty < u < \infty$, then

$$A^{u+t} = A^u A^t$$

and so it is easy to see that

$$B_n^{(t+u)}(x^k) = B_n^{(t)}(B_n^{(u)}(x^k)) = B_n^{(u)}(B_n^{(t)}(x^k)),$$

and hence

$$B_n^{(t+u)}(p) = B_n^{(t)}(B_n^{(u)}(p)) = B_n^{(u)}(B_n^{(t)}(p))$$

for any polynomial p of degree at most n .

If f is bounded on $[0, 1]$, we can now define

$$(4.8) \quad B_n^{(t)}(f) = B_n^{t-1}(B_n(f)).$$

This definition focuses attention on the case $t = 0$. The polynomial

of degree at most n

$$B_n^*(f) = B_n^{(0)}(f) = B_n^{-1}(B_n f)$$

is a kind of surrogate f . How is this polynomial related to f ? It is clear that if $f = p$, a polynomial of degree at most n , then

$$B_n^* p = p .$$

In particular, let $p = L_n(f)$ be the unique polynomial of degree at most n which agrees with $f(x)$ at $x = j/n, j = 0, \dots, n$. Then $B_n(f) = B_n(L_n(f))$ and so

$$B_n^*(f) = B_n^*(L_n(f)) = L_n(f) .$$

Of course, this result could have been obtained without the apparatus of this paper, but it comes out of our discussion quite naturally.

We wish to thank Benjamin Weiss for some helpful advice on this work.

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