ON THE UNION OF TWO STARSHAPED SETS

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Let $S$ be a compact subset of a topological linear space. We shall say that $S$ has the property $\varphi$ if there exists a line segment $R$ such that each triple of points $x, y$ and $z$ in $S$ determines at least one point $p$ of $R$ (depending on $x, y$ and $z$) such that at least two of the segments $xp, yp$ and $zp$ are in $S$. It is clear that if $S$ is the union of two starshaped sets then $S$ has the property $\varphi$, and the problem has been raised by F. A. Valentine [1] as to whether the property $\psi$ ensures that $S$ is the union of two starshaped sets. We shall show that this is not so, in general, but we begin by giving a further constraint which ensures the result.

**Theorem.** If a compact set $S$, of a topological linear space, has the property $\varphi$, and, for any point $q$ of $S$, the set of points of $R$ which can be seen, via $S$, from $q$ form an interval, then $S$ is the union of two starshaped sets.

**Proof.** Consider the collection of sets $\{T_q\}, q \in S$, where $T_q$ denotes the set of points of $R$ which can be seen, via $S$, from $q$. If every two intervals of this collection have a nonempty intersection, then it follows from Helly's Theorem that $S$ is starshaped from a point of $R$. Suppose, therefore, that there exist points $q_1, q_2$ of $S$ such that $T_{q_1} \cap T_{q_2} = \emptyset$. We partition the collection $\{T_q\}, q \in S$, into three collections $\{T_q\}_1, \{T_q\}_2, \{T_q\}_12$, so that $T_q$ belongs to $\{T_q\}_1$ if $T_q$ meets $T_{q_1}$ but not $T_{q_2}$, $T_q$ belongs to $\{T_q\}_2$ if $T_q$ meets $T_{q_2}$ but not $T_{q_1}$, $T_q$ belongs to $\{T_q\}_12$ if $T_q$ meets both $T_{q_1}$ and $T_{q_2}$. If $T_q, T_r$ are two sets of $\{T_q\}_1, (i = 1, 2)$ then it follows from $\varphi$ applied to the points $q, r$ and $q_j (j \neq i)$ that $T_q$ meets $T_r$. If $T_q, T_r$ are two sets of $\{T_q\}_2$, then, since both $T_q$ and $T_r$ span the gap between $T_{q_1}$ and $T_{q_2}$, it follows that $T_q$ meets $T_r$. Further, if $T_q$ belongs to $\{T_q\}_12$, then it must meet every set of at least one of the collections $\{T_q\}_i, (i = 1, 2)$. For, otherwise, there exists sets $T_{r_1}, T_{r_2}$, belonging to $\{T_q\}_1, \{T_q\}_2$ respectively, which do not meet $T_q$. However, by property $\varphi$ applied to $r_1, r_2$ and $q$, this implies that $T_{r_1}$ meets $T_{r_2}$ and hence that

$$T_{r_1} \cup T_{r_2}$$

spans the gap between $T_{q_1}$ and $T_{q_2}$. But this implies that $T_{r_1} \cup T_{r_2}$ meets $T_q$; contradiction. We now form the collections $\{T_q\}_{12}(i = 1, 2)$ so that $T_q$ belongs to $\{T_q\}_{12}$ if either $T_q$ is in the collection $\{T_q\}_i$ or $T_q$ is in $\{T_q\}_{12}$ and meets every member of $\{T_q\}_i$. We note that
and combining the results above with Helly’s Theorem, we deduce that the intersection $U_i$ of all the members of $\{T_q\}_{i=1}^n$ is a nonempty closed set. Let $s_i$ be a point of $U_i$ and let $S_i$ be the set of points of $S$ which can be seen, via $S$, from $S_i$. Then $S$ is the union of $S_1$ and $S_2$ which are starshaped from $s_1$ and $s_2$ respectively.

**COUNTER-EXAMPLE.** There exists a plane compact set $S$ which has the property $\varphi$ but, nevertheless, cannot be expressed as the union of two starshaped sets.

We assume the existence of a rectangular coordinate system and let $c, e, v$ be the vertices of an equilateral triangle, with $c, e$ on the $x$-axis, $e$ lying to the right of $c$, and $v$ lying above the $x$-axis. Let $o$ be the centroid of the triangle $cev$ and let the line through $o$, which is parallel to the $x$-axis, meet $cv, ev$ at $g, h$ respectively. Let the vertical line through $g$ meet $co$ at $i$ and $ce$ at $k$. Let the vertical line through $h$ meet $eo$ at $j$ and $ce$ at $\rho$. Let $vi$ produced meet $ck$ at $a$ and let $vj$ produced meet $pe$ at $b$. So far we have defined six distinct points $c, a, k, \rho, b, e$, in that order, on the $x$-axis. Let $d$ be a point on the $x$-axis which lies to the left of $e$ and let the line $od$ produced meet $cg$ at $m$ and $hv$ at $n$. Suppose the lines $mi$ produced, $nj$ produced, meet the $x$-axis at points $k', \rho'$, respectively. Let $kg$ meet $mb$ at $v_2$ and let $pi$ produced meet $am$ at $v_3$. As 

$$d \to -\infty, \rho' \to \rho, k' \to k, m \to g, v_1 \to g.$$ 

Hence we can suppose that $d$ has been chosen as to ensure that (i)
$k'$ and $\rho'$ are distinct interior points of $ab$, with $k'$ lying to the left
of $\rho'$, and (ii) the quadrilateral $mv_1iv$ is nondegenerate, and $i$ is
closer to the $x$-axis than is $m$. We choose a point $f$ on the $x$-axis
and to the right of $e$, and a point $w$ on the line $ov$ produced and
strictly above $v$. Let $ev$ produced meet $dw$ at $p$ and let $cv$ produced
meet $wf$ at $z$. We also choose a point $q$ on $vo$ produced, which lies
strictly above the $x$-axis but which lies below the line segments $a|j$
and $bi$. Now, by (ii), the interior $C_1$ of the quadrilateral $mv_1iv$ is
nonempty, and, if $kj$ produced meets $nb$ at $u$, the interior $C_2$ of
the triangle $jun$ is nonempty. We define $C_3$ to be the interior of the
triangle $aqb$ together with the open line segment $ab$. Finally we
take $S$ to be $T - C_1 \cup C_2 \cup C_3$, where $T$ denotes the set within and
on $wdf$. Note that by construction every point of $S$, other than those
within $vzwp$, can see, via $S$, one of $a$ and $b$, and one of $c, d$ and $e$.
We first show that $S$ has the property $\phi$, with $R = df$.

Suppose that $p_1, p_2, p_3$ are points of $S$ for which no two can toge-
ther be seen from any point of $df$. As any point within $vzwp$ can
see each of $c, d$ and $e$, we deduce, from above, that none of $p_1, p_2, p_3$
can lie within $vzwp$. But this implies that each of $p_1, p_2, p_3$ can see
one of $a$ and $b$; contradiction. Therefore, we conclude that such a
triple of points cannot be chosen in $S$ and hence that $S$ has the pro-
erty $\phi$, with $R = df$.

We now show that $S$ is not the union of two starshaped sets.
Suppose, therefore, that $p_1, p_3$ are points of $S$ and that each point of
$S$ can be seen from at least one of $p_1, p_3$. Let $am$ produced meet
$dw$ at $a'$ and let $bn$ produced meet $wf$ at $b'$. If neither of $p_1, p_2$ lie
within $aa'd$, then neither point can see the interior of the segment
$mv_1$. Hence $p_1$, say, lies within $aa'd$ and, similarly, $p_3$ lies within $bb'f$.
Let $iv_2$ produced meet $dw$ at $i'$ and let $ju$ produced meet $wf$ at $j'$.
Then $p_1$ must lie within $dv_2v'$, for, otherwise, the interior of the line
segment $v_1i$ cannot be seen from $p_1$ or $p_3$. Similarly $p_3$ lies within
$fbuj'$. Let $ni$ produced meet the $x$-axis at $n'$ and let $mj$ produced
meet the $x$-axis at $m'$. As $p_2$ cannot see the interior of the line seg-
ment $nj, p_1$ must lie within $a'n'$. But then $p_1$ cannot see the interior
of the line segment $mv_1$ and so $p_2$ must lie within $jbm'$. We note
that $p_1 = a$, $p_2 = b$ is impossible and that $i$ and $j$ are the same
distance from the $x$-axis. It follows that $p_1 i$ produced, $p_2 j$ produced meet
at an interior point $g'$ of $ijv$. But as $C_1$ and $C_2$ are nonempty open
sets, it follows that there is a nonempty quadrilateral $Q$, which lies
within $ijv$ and has $g'$ as its lowest vertex, whose interior cannot be
seen from either $p_1$ or $p_3$. As $Q$ lies in $S$, this is a contradiction, and
we conclude that $S$ cannot be expressed as the union of two starshaped
sets.
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