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A set of points M is said to cover a graph G if every line in G has at least one point in M. Call M a minimum cover (m.c.) for G if it is a point cover with a minimum number of points. The number of points in any minimum cover of a graph G is called the point covering number of G and is denoted by $\alpha(G)$. If x is a line in G, denote by $G - x$ the graph obtained from G by deleting x. Similarly, if v is a point of G, $G - v$ will denote the graph obtained from G by deleting v and all lines incident with v. A line x in G is said to be a critical line (with respect to point covering) if $\alpha(G - x) < \alpha(G)$. A graph is called line-critical if every line is critical. Obviously every complete graph is line-critical, and so is any odd cycle. There are, however, many other line-critical graphs.

The main purpose of this paper is to prove that in any graph, two adjacent critical lines must lie on an odd cycle. This result will imply that a line-critical graph must be a block and furthermore, that any two adjacent lines in such a graph must lie on an odd cycle.

This condition, though necessary for a graph to be line-critical, is not sufficient as is illustrated by the graph in Figure 1.

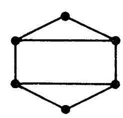

FIGURE 1.

The concept of a line-critical graph is introduced by Ore [4] and briefly considered by Erdös and Gallai [3]. However, a structural characterization of line-critical graphs remains unknown, although one of the authors [5] constructs an infinite family of such graphs which, in particular, includes all line-critical graphs with fewer than eight points.

2. **Additional terminology.** A *graph* G consists of a finite nonempty set of points $V(G)$ and a set of lines $E(G)$ each of which is an unordered pair of points. The line uv is *incident* with each of its points. Two points (lines) joined by a line (point) are said to be *adjacent*. Let $|A|$ denote the number of elements in the set A. The *degree* of a point v is the number of lines incident with it and is denoted by $d(v)$. A line is called an *endline* if one of its points has

degree one. A point v is *isolated* if $d(v) = 0$.

A *walk* W of G is an alternating sequence of points and lines, beginning and ending with points (said to be *joined* by W) such that each line is incident with the points before and after it. A walk P is a *path* if its points are distinct. The *length* of a walk is the number of occurrences of lines in it. If W is a walk beginning and ending with the same point, W is said to be *closed*. A closed walk in which all points are distinct is called a *cycle*.

The graph G is *connected* if every two points are joined by a path. A point v is a *cutpoint* of the connected graph G if $G - v$ is disconnected. If v is any point of a graph G, a *branch* of G at v is a maximal connected subgraph of G not having v as a cutpoint. A connected graph is a *block* if it has no cutpoints. If G is a graph and H is a subgraph of G, then $G - H$ is the subgraph of G defined by (1) $V(G - H) = V(G)$ and (2) $E(G - H) = E(G) - E(H)$.

3. Preliminary results.

The main result of this paper will be obtained in the following manner. First it will be shown that any two critical lines which are adjacent must lie on a cycle. Next in Theorem 2 it is proved that if the graph is bipartite, then no two critical lines are adjacent. Then based on the assumption that every cycle containing two adjacent critical lines is even, a contradiction to Theorem 2 is obtained. Finally, it is pointed out that since the point covering number is additive on the components of a graph, it may be assumed without loss of generality that all graphs are connected.

THEOREM 1. *Any two adjacent critical lines of a graph lie on a cycle.*

Proof. Suppose the conclusion is false, so that there are adjacent critical lines x and y in G which do not lie on a common cycle. Let $x = uw$ and $y = vw$. Then w is a cutpoint of G. Partition the lines of G as follows. Let H be the branch of G at w which contains x and let J be the branch containing y. Hence G has the general appearance of Figure 2.

Let M be an m.c. for $G - x - y$. Then $M \cup \{w\}$ covers G. Thus $\alpha(G - x - y) = \alpha(G) - 1$, and hence $\alpha(G - x) = \alpha(G - x - y) = \alpha(G - y)$.

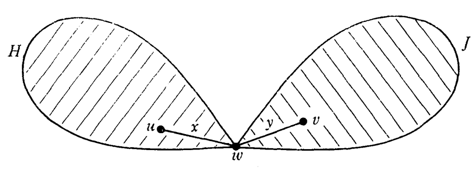

FIGURE 2.

Thus every m.c. for $G - x$ and every m.c. for $G - y$ is also an m.c. for $G - x - y$. Let M_x be an m.c. for $G - x$. Hence $u, w \notin M_x$ and $v \in M_x$. Similarly if M_y is an m.c. for $G - y$, then $u \in M_y$, and $v, w \notin M_y$.

Next it is shown that $M_x \cap V(J)$ is an m.c. for J by assuming the contrary. Then there is a set $N \subset V(J)$ which covers J but with $|N| < |M_x \cap V(J)|$. Hence $W = N \cup [M_x \cap V(H)] \cup \{w\}$ covers G and it follows that

$$\begin{aligned}\alpha(G) &\leqq |W| = |N \cup [M_x \cap V(H)] \cup \{w\}| = |N \cup \{w\}| + |M_x \cap V(H)| \\ &\leqq |N| + 1 + |M_x \cap V(H)| \leqq |M_x \cap V(J)| + |M_x \cap V(H)| \\ &= |M_x| = \alpha(G - x)\,,\end{aligned}$$

thus contradicting the hypothesis that x is critical in G. Similarly, $M_y \cap V(H)$ is an m.c. for H.

Next it will be shown that $R = [M_x \cap V(J)] \cup [M_y \cap V(H)]$ is an m.c. for $G - x - y$. Clearly R covers $G - x - y$. Let S be any m.c. for $G - x - y$. Now if $w \in S$, then S covers G and hence $\alpha(G) \leqq \alpha(G - x - y)$ which is a contradiction. Hence $w \notin S$ and thus $S \cap V(H)$ and $S \cap V(J)$ are disjoint sets. Now it is shown that $S \cap V(H)$ is an m.c. for H by assuming the contrary. Then there is a set U which covers H with $|U| < |S \cap V(H)|$. Now $w \in U$ or else S is not an m.c. for $G - x - y$. Hence $U \cup [S \cap V(J)]$ covers G. Thus

$$\begin{aligned}\alpha(G) &\leqq |U| + |S \cap V(J)| < |S \cap V(H)| + |S \cap V(J)| \\ &= |S| = \alpha(G - x - y)\end{aligned}$$

which is absurd. Thus $S \cap V(H)$ is an m.c. for H. Similarly, $S \cap V(J)$ is an m.c. for J. Hence

$$\begin{aligned}|S| &= |S \cap V(H)| + |S \cap V(J)| = \alpha(H) + \alpha(J) \\ &= |M_x \cap V(J)| + |M_y \cap V(H)| = |R|\end{aligned}$$

and hence R is an m.c. for $G - x - y$. But $u \in M_y \cap V(H)$ and $v \in M_x \cap V(J)$, hence $u, v \in R$. Thus R covers G and $\alpha(G) \leqq |R| = \alpha(G - x - y)$ again contradicting the hypothesis that x is critical in G. This completes the proof of the theorem.

The next theorem is a corollary to the results obtained by Dulmage and Mendelsohn [1, 2]. However, since a direct proof is short, it is included for the sake of completeness.

THEOREM 2. *No two critical lines of a bipartite graph are adjacent.*

Proof. Let G be a bipartite graph and suppose G contains two adjacent critical lines $x = uw$ and $y = wv$. Let the two point sets of of G be S and T, and let M_x and M_y be m.c.'s for $G - x$ and $G - y$ respectively. Let $S_x = S \cap M_x$, $S_y = S \cap M_y$, $T_x = T \cap M_x$, and $T_y = T \cap M_y$. It may be assumed without loss of generality that $u, v \in S$ and $w \in T$. Now $u, w \notin M_x$, but $v \in M_x$, while $v, w \notin M_y$, but $u \in M_y$.

Since $M_x \cup \{w\}$ and $M_y \cup \{w\}$ both cover G, $\alpha(G - x) = \alpha(G - y) = \alpha(G - x - y)$. Thus M_x and M_y are both m.c.'s for $G - x - y$. But then by Theorem 5 of [1], $N = S_x \cup S_y \cup (T_x \cap T_y)$ is also an m.c. for $G - x - y$. Thus N contains both u and v and hence N covers G. Therefore $\alpha(G) \leqq \alpha(G - x - y) = \alpha(G - x)$, contradicting the hypothesis that x is critical in G, and thus completing the proof.

4. The main theorem. The principal result of this paper may now be proved.

THEOREM 3. *Every two adjacent critical lines of a graph lie on an odd cycle.*

Proof. Let $x = uw$ and $y = vw$ be adjacent critical lines in G. By Theorem 1, there is at least one cycle in G containing both x and y. Consider all cycles of G containing both lines x and y. To prove the theorem, assume that all such cycles are even.

ASSERTION 1. *If H is the subgraph of G induced by all the lines of the cycles containing both x and y, other than x and y themselves, then H is bipartite.*

To prove this assertion, let C be a cycle in H. Assume C to be odd. Note that $w \notin V(C)$, for if $w \in V(C)$, then C contains a line incident with w. But such a line cannot lie on a cycle with x and y. Let z be a line of C. By definition of H, z must lie on a cycle C' containing x and y. Let P_1' and P_2' be the paths traversed along C' by starting at u and v respectively, and stopping upon encountering a point of C for the first time. Let $v_1 = V(C) \cap V(P_1')$ and $v_2 = V(C) \cap V(P_2')$. Now $v_1 \neq v_2$ since C' contains z. Note it may be that $v_1 = u$ or $v_2 = v$. The points v_1 and v_2 thus induce a division of C into two paths Q_1 and Q_2 each of which contains at least one line. But then one of the two cycles $P_1' \cup P_2' \cup Q_1 \cup \{x, y\}$ and $P_1' \cup P_2' \cup Q_2 \cup \{x, y\}$ must be of odd length, contradicting the original assumption, and completing the proof of Assertion 1.

Let $A = \{r_1, \cdots, r_m\}$ and $B = \{s_1, \cdots, s_n\}$ be the point sets of the bipartite graph H with $u, v \in B$.

Now consider the graph $J = G - H$. The graph J can be partitioned into maximal subgraphs (some perhaps containing no lines), $R_1, \cdots R_m$, $S_1, \cdots S_n$, and T with $r_i \in V(R_i)$, $s_j \in V(S_j)$, and $w \in V(T)$

so that the intersection of any two of these is at most the point w. To see this, it need only be shown that any path joining points of H is contained in H unless it contains w. This is now proved:

ASSERTION 2. *If u_0 and u_1 are distinct points of H and if P is a path joining them which does not include w, then all lines of P are in H.*

Let $Q = H + x + y$. Then, clearly, Q is a block and hence $Q \cup P$ is a block. Let z be a line of P. Then z and w lie on a common cycle. But the only lines of Q which meet w are x and y. Hence z, x, and y lie on a common cycle and $z \in H$, proving the assertion.

Thus the graph $G = H \cup J$ has the appearance of Figure 3.

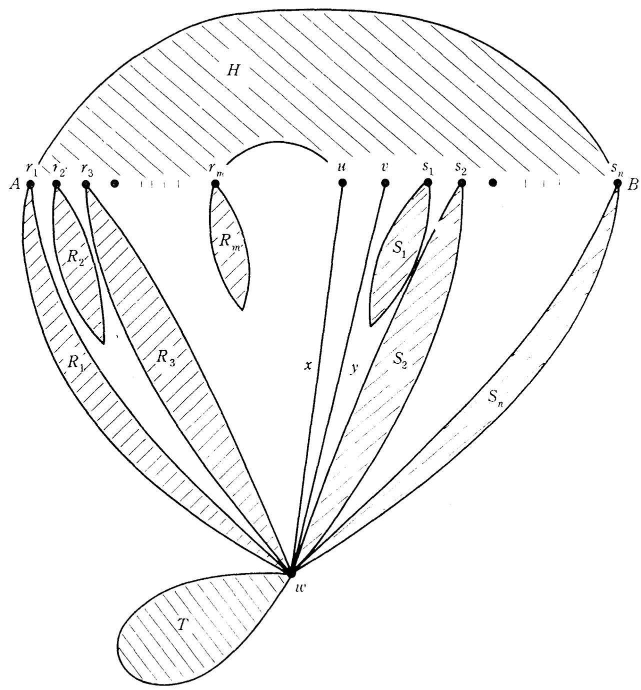

FIGURE 3.

The next objective is to show that m.c.'s for $G - x$ and $G - y$ can be found which meet $V(G) - V(H)$ in exactly the same points. Let X and Y be m.c.'s for $G - x$ and $G - y$ respectively. Hence $u \in Y - X$, $v \in X - Y$, and $w \notin X \cup Y$. Next note that if there is a point $v' \in X \cap Y$, then $X - v'$ and $Y - v'$ cover $G - x - v'$ and $G - y - v'$ respectively and hence x and y are both critical in $G - v'$.

Now since $w \notin X \cup Y$, both $X \cap V(T)$ and $Y \cap V(T)$ cover T. Furthermore, they have the same number of points. Hence it may be assumed that they are identical.

Next, consider the two sets $X_i = X \cap V(R_i)$ and $Y_i = Y \cap V(R_i)$ for any i. There are exactly three possibilities.

Case 1. Suppose $r_i \notin X_i \cup Y_i$. Then by the same argument as for T, R_i is covered by both X_i and Y_i which must have the same number of points. Hence they may be taken to be equal.

Case 2. Similarly, if $r_i \in X_i \cap Y_i$ it may be assumed that

$$[V(R_i) - \{r_i\}] \cap X = [V(R_i) - \{r_i\}] \cap Y .$$

Case 3. Finally, suppose $r_i \in X_i - Y_i$. For this case the following statement is required.

ASSERTION 3. *If*

$$X_i = X \cap V(R_i) , \quad Y_i = Y \cap V(R_i) , \quad r_i = V(H) \cap V(R_i) ,$$

and if $r_i \in X_i - Y_i$, *then* $|Y_i| \leqq |X_i| \leqq |Y_i| + 1$.

Suppose first that $|X_i| < |Y_i|$. Then replace Y_i in Y with X_i. A cover for $G - y$ with fewer elements than Y is thus obtained, contradicting the minimality of Y. Hence $|Y_i| \leqq |X_i|$.

Next suppose that $|Y_i| + 1 < |X_i|$. Then $|Y_i| < |X_i| - 1$ and thus $|Y_i \cup \{r_i\}| < |X_i|$. But then $[X - X_i] \cup [Y_i \cup \{r_i\}]$ covers $G - x$ and has fewer elements than X, contradicting the minimality of X. Hence $|X_i| \leqq |Y_i| + 1$ and the assertion is verified.

Now if $|Y_i| = |X_i|$, X_i and Y_i may be assumed to be identical as in Case 1. Then it may be assumed that $r_i \in X_i \cap Y_i$. If $|X_i| > |Y_i|$, then X_i may be taken to be $Y_i \cup \{r_i\}$.

The results of Cases 1, 2, and 3 thus allow the assumption that the points in the symmetric difference $(X - Y) \cup (Y - X)$ are in H, i.e., are in $A \cup B$. Now form a new graph $G' = G - [(X \cup Y) - (A \cup B)]$. The definition of G' allows the presence of isolated points. However, since such points have nothing to do with the point covers of the graph, they will be regarded as deleted. Thus G' consists of H together with possibly some lines incident with an r_i or an s_j. Each of these lines is either an endline or is incident with w.

The graph G' thus has the appearance of Figure 4.

Now clearly x and y lie only on even cycles in G'. It was pointed out above that if $v' \in X \cap Y$, where X and Y are m.c.'s for $G - x$ and $G - y$ respectively, and if x and y are critical in G, then x and y remain critical in $G - v'$. By repeated application of this property, it follows that x and y are critical in G'.

Next, build a new graph G'' from G' (cf. Figure 5) by splitting

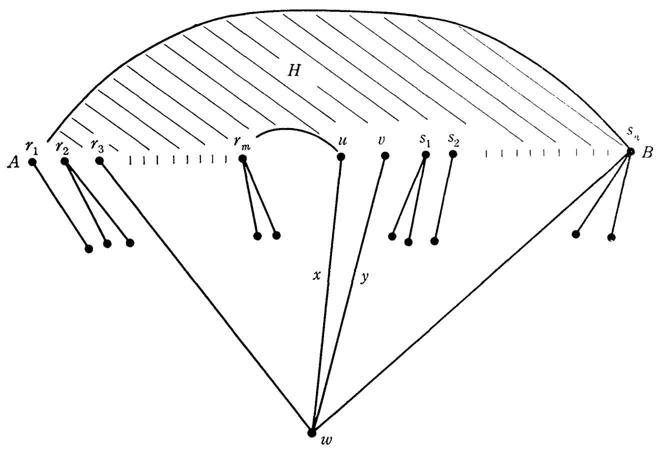

FIGURE 4.

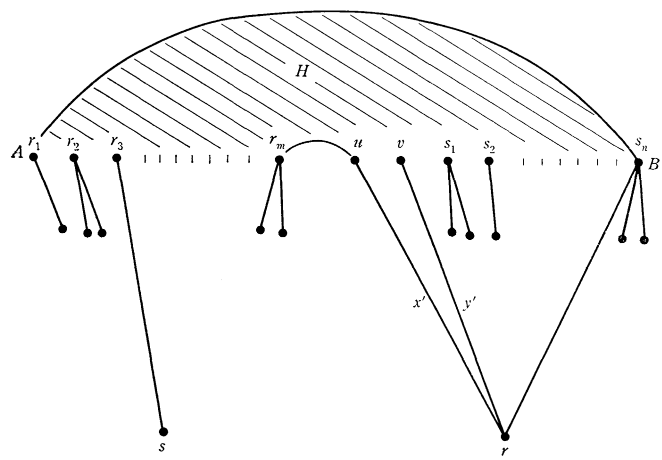

FIGURE 5.

w into two points r and s such that r is adjacent to those points s_j and s to those points r_i to which w was adjacent.

In G'' let $x' = ur$ and $y' = vr$. It is immediate that x' and y' are critical in G''. But the new graph G'' is bipartite. Hence Theorem 2 is contradicted and the proof of Theorem 3 is complete.

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