

# Pacific Journal of Mathematics

**ABELIAN  $p$ -GROUPS DETERMINED BY THEIR ULM  
SEQUENCES**

PETER CRAWLEY

## ABELIAN $p$ -GROUPS DETERMINED BY THEIR ULM SEQUENCES

PETER CRAWLEY

**Ulm's theorem asserts that, within the class of all reduced countable abelian  $p$ -groups, a group is determined, up to isomorphism, by its Ulm sequence. Although this theorem fails in general for uncountable groups, there are classes of uncountable abelian  $p$ -groups whose members are determined within the class by their Ulm sequences. Kolettis has shown that the class of direct sums of countable  $p$ -groups has this property. Here it is shown that the class of those abelian  $p$ -groups for which the Ulm type is finite and all the Ulm factors except the last are direct sums of cyclic groups, is another such class.**

Let  $G$  be a reduced abelian  $p$ -group. Define the subgroup  $G^\alpha$  for each ordinal  $\alpha$  as follows. Set  $G^0 = G$ . Proceeding inductively, if  $\alpha = \beta + 1$ , define  $G^\alpha$  to be the subgroup of those elements in  $G^\beta$  which have infinite height in  $G^\beta$ ; if  $\alpha$  is a limit ordinal, define  $G^\alpha = \bigcap_{\beta < \alpha} G^\beta$ . Since  $G$  is reduced, there is a first ordinal  $\tau$  such that  $G^\tau = 0$ ; this ordinal  $\tau$  is called the *Ulm type* of  $G$ . The *Ulm factors* of  $G$  are defined to be the factor groups  $G_\alpha = G^\alpha/G^{\alpha+1}$  ( $\alpha > \tau$ ). And the sequence of Ulm factors  $G_\alpha$  ( $\alpha < \tau$ ) is called the *Ulm sequence* of  $G$ . Two groups  $G$  and  $H$  have *isomorphic Ulm sequences* if  $G_\alpha \cong H_\alpha$  for every  $\alpha$ .

Our theorem is now the following:<sup>1</sup>

*If  $G$  is a reduced abelian  $p$ -group having finite Ulm type  $n$  and such that its first  $n - 1$  Ulm factors  $G_0, \dots, G_{n-2}$  are direct sums of cyclic groups, and if  $H$  is any other abelian  $p$ -group whose Ulm sequence is isomorphic to that of  $G$ , then  $H \cong G$ .*

It should be noted that in this theorem no assumption is made on the last Ulm factor  $G_{n-1}$  of  $G$ .

Neither the assumption of finite Ulm type nor the assumption that all but the last Ulm factor are direct sums of cyclic groups can be dropped from the hypotheses of the theorem. For example, if  $G$  is any countable  $p$ -group whose Ulm type is infinite, then there is an uncountable  $p$ -group whose Ulm sequence is isomorphic to that of  $G$ . Moreover, it is known that there are nonisomorphic  $p$ -groups of Ulm type 2 having isomorphic Ulm sequences.

---

<sup>1</sup> Since completing this paper, I have learned that P. Hill and C. Megibben have obtained similar results.

Throughout the proof of the theorem, the usual notation and terminology, for the most part, is used.<sup>2</sup> If  $x$  is an element of a group, then the cyclic subgroup generated by  $x$  is denoted by  $[x]$ . The symbol  $\oplus$  will be used to designate the direct sum of a pair of subgroups, while the symbol  $\sum$  will designate the direct sum of a family of subgroups.

The proof begins with the following common generalization of Kolettis' theorem [5] and a theorem of Zippin [6].

(a) *Let  $G$  and  $H$  be reduced abelian  $p$ -groups which are direct sums of countable groups, and whose Ulm sequences are isomorphic. If  $\sigma$  is an ordinal, and  $f$  is an isomorphism of  $G^\sigma$  onto  $H^\sigma$ , then  $f$  extends to an isomorphism of  $G$  onto  $H$ .*

*Proof.* Write  $G = \sum_{i \in I} G_i$  and  $H = \sum_{j \in J} H_j$  where each  $G_i$  and each  $H_j$  is countable. For every subset  $K \subseteq I$  and every subset  $L \subseteq J$  set  $G(K) = \sum_{i \in K} G_i$  and  $H(L) = \sum_{j \in L} H_j$ .

Let  $\alpha$  be an ordinal less than  $\sigma$ . For each  $i \in I$  and each  $j \in J$  pick sequences of elements  $\{b_{i,\alpha,n}\}_{n < \infty}$  in  $G_i^\alpha$  and  $\{c_{j,\alpha,m}\}_{m < \infty}$  in  $H_j^\alpha$  such that

$$G_i^\alpha / G_i^{\alpha+1} = \sum_{n < \infty} [b_{i,\alpha,n} + G_i^{\alpha+1}]$$

and

$$H_j^\alpha / H_j^{\alpha+1} = \sum_{m < \infty} [c_{j,\alpha,m} + H_j^{\alpha+1}].$$

Then

$$\begin{aligned} G^\alpha / G^{\alpha+1} &= \sum_{i,n} [b_{i,\alpha,n} + G^{\alpha+1}] \\ &\cong H^\alpha / H^{\alpha+1} = \sum_{j,m} [c_{j,\alpha,m} + H^{\alpha+1}], \end{aligned}$$

and consequently there exists a one-to-one function  $\phi_\alpha$  which maps the set  $\{b_{i,\alpha,n} \mid i \in I; n = 1, 2, \dots\}$  onto the set  $\{c_{j,\alpha,m} \mid j \in J; m = 1, 2, \dots\}$  in such a way that  $b_{i,\alpha,n} + G^{\alpha+1}$  and  $\phi_\alpha(b_{i,\alpha,n}) + H^{\alpha+1}$  have the same order.

We now construct two sequences of subsets  $I_\nu \subseteq I$  and  $J_\nu \subseteq J$  such that the following conditions hold for each ordinal  $\nu$ :

- (i)  $I_\nu$  and  $J_\nu$  are countable and nonempty.
- (ii)  $I_\nu \cap I_\mu = J_\nu \cap J_\mu = \emptyset$  for  $\mu < \nu$ .

Let  $I^\nu$  denote the set-union of the  $I_\mu$  ( $\mu < \nu$ ), and let  $J^\nu$  denote the set-union of the  $J_\mu$  ( $\mu < \nu$ ).

---

<sup>2</sup> See, for example, Fuchs [2] or Kaplansky [4].

(iii) If  $i \in I_\nu$  and  $x \in G_i^\sigma$ , then  $f(x) \in H(J^{\nu+1})$ . Conversely, if  $j \in J_\nu$  and  $y \in H_j^\sigma$ , then  $f^{-1}(y) \in G(I^{\nu+1})$ .

(iv) If  $i \in I_\nu$ , then  $\Phi_\alpha(b_{i,\alpha,n}) \in H(J_\nu)$  for all  $\alpha$  and  $n$ . Conversely, if  $j \in J_\nu$ , then  $\Phi_\alpha^{-1}(c_{j,\alpha,m}) \in G(I_\nu)$  for all  $\alpha$  and  $m$ .

Suppose the sets  $I_\mu$  and  $J_\mu$  have been constructed for each  $\mu < \nu$ , and suppose that  $I^\nu \neq I$ . Pick two sequences of countable subsets  $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots \subseteq I - I^\nu$  and  $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots \subseteq J - J^\nu$  such that  $M_1 \neq \emptyset$  and such that the following hold for each positive integer  $q$ :

$$\begin{aligned} f(G(M_q)^\sigma) &\subseteq H(N_q) \oplus H(J^\nu), \\ \Phi_\alpha(b_{i,\alpha,n}) &\in H(N_q) && \text{if } i \in M_q, \\ f^{-1}(H(N_q)^\sigma) &\subseteq G(M_{q+1}) \oplus G(I^\nu), \\ \Phi_\alpha^{-1}(c_{j,\alpha,m}) &\in G(M_{q+1}) && \text{if } j \in N_q. \end{aligned}$$

Then if  $I_\nu = M_1 \cup M_2 \cup M_3 \cup \dots$  and  $J_\nu = N_1 \cup N_2 \cup N_3 \cup \dots$ , it is clear that  $I_\nu$  and  $J_\nu$  satisfy (i)-(iv).

Since all the sets  $I_\nu$  are nonempty, there exists an ordinal  $\lambda$  such that  $I^\lambda = I$  and  $J^\lambda = J$ . Consequently,  $G = G(I^\lambda)$  and  $H = H(J^\lambda)$ , and for each  $\nu < \lambda$ ,

$$(1) \quad G(I^{\nu+1}) = G(I^\nu) \oplus G(I_\nu) \quad \text{and} \quad H(J^{\nu+1}) = H(J^\nu) \oplus H(J_\nu).$$

Suppose that there is an isomorphism  $g_\nu$  of  $G(I^\nu)$  onto  $H(J^\nu)$  such that the restrictions of  $g_\nu$  and  $f$  to  $G(I^\nu)^\sigma$  are equal. Let  $f_\nu$  denote the restriction of  $f$  to  $G(I_\nu)^\sigma$ , and let  $\pi_0$  denote the projection of  $G(I^{\nu+1})$  onto  $G(I_\nu)$ ,  $\pi_1$  denote the projection of  $H(J^{\nu+1})$  onto  $H(J_\nu)$ , and  $\pi_2$  denote the projection of  $H(J^{\nu+1})$  onto  $H(J^\nu)$ , as determined by the decompositions (1). If  $x \in G(I_\nu)^\sigma$ , then by (iii) there exist elements  $y \in H(J_\nu)$  and  $z \in H(J^\nu)$  such that  $f(x) = y + z$ . Then  $f^{-1}(y) = x - f^{-1}(z)$ , and as  $f^{-1}(z) \in G(I^\nu)$ , it follows that  $\pi_0 f^{-1} \pi_1 f(x) = x$ . Similarly, if  $y \in H(J_\nu)^\sigma$ , then  $\pi_1 f \pi_0 f^{-1}(y) = y$ . Therefore  $\pi_1 f_\nu$  is an isomorphism of  $G(I_\nu)^\sigma$  onto  $H(J_\nu)^\sigma$ . Moreover, for each  $\alpha < \sigma$ , the  $\alpha$ th Ulm factors of  $G(I_\nu)$  and  $H(J_\nu)$  are isomorphic by (iv), and hence  $G(I_\nu) \cong H(J_\nu)$  by Ulm's theorem. By Zippin's theorem [6, § 8], the mapping  $\pi_1 f_\nu$  extends to an isomorphism  $h_1$  of  $G(I_\nu)$  onto  $H(J_\nu)$ . And inasmuch as  $\pi_2 f_\nu$  is a homomorphism of  $G(I_\nu)^\sigma$  into  $H(J^\nu)^\sigma$ , this homomorphism extends to a homomorphism  $h_2$  of  $G(I_\nu)$  into  $H(J^\nu)$  by [1, 1.2]. Consequently, if  $h = h_1 + h_2$ , and if  $g_{\nu+1}$  is the direct sum of  $g_\nu$  and  $h$ , then  $g_{\nu+1}$  is an isomorphism of  $G(I^{\nu+1})$  onto  $H(JI^{\nu+1})$  whose restriction to  $G(I^{\nu+1})^\sigma$  equals the restriction of  $f$  to  $G(I^{\nu+1})^\sigma$ . A transfinite induction now completes the proof of (a).

(b) Let  $G$  be a reduced abelian  $p$ -group, and let  $B$  be a basic

subgroup of  $G^1$ . Then there is a subgroup  $S$  such that  $S + G^1 = G$  and  $S^1 = S \cap G^1 = B$ .

*Proof.* Since  $G^1/B$  is divisible, it is a direct summand of  $G/B$ , say  $G/B = G^1/B \oplus S/B$ , where  $S$  is a subgroup of  $G$  containing  $B$ . Then  $S + G^1 = G$ , and  $S \cap G^1 = B$ . Suppose  $b \in B$ , and  $n$  is any positive integer. Then as  $b \in G^1$ , there are elements  $c \in S$  and  $u \in G^1$  such that  $b = p^n c + p^n u$ . Therefore  $p^n c \in G^1$ , and consequently  $p^n c \in B$ . Hence  $p^n u \in B$ , and inasmuch as  $B$  is pure in  $G^1$ , there is an element  $d \in B$  such that  $p^n u = p^n d$ . Thus  $b = p^n(c + d)$ , and it follows that  $B \subseteq S^1 \subseteq S \cap G^1$ .

The next lemma is a special case of the principal result of Irwin-Richman [3].

(c) *If  $G$  is an abelian  $p$ -group for which  $G/G^1$  and  $G^1$  are direct sums of cyclic groups, then  $G$  is a direct sum of countable groups.*

(d) *If  $G$  and  $H$  are reduced abelian  $p$ -groups such that  $G/G^1 \cong H/H^1$  and  $G/G^1$  is a direct sum of cyclic groups, and if  $f$  is an isomorphism of  $G^1$  onto  $H^1$ , then  $f$  extends to an isomorphism of  $G$  onto  $H$ .*

*Proof.* Let  $B$  be a basic subgroup of  $G^1$ , and let  $C$  be that basic subgroup of  $H^1$  which is the image of  $B$  under  $f$ . By (b) there are subgroups  $S$  of  $G$  and  $T$  of  $H$  such that  $S + G^1 = G$ ,  $S \cap G^1 = S^1 = B$ ,  $T + H^1 = H$ , and  $T \cap H^1 = T^1 = C$ . Now

$$S/S^1 = S/B \cong G/G^1 \cong H/H^1 \cong T/C = T/T^1,$$

and hence by (c) and (a) there is an isomorphism  $\varphi$  of  $S$  onto  $T$  such that the restriction of  $\varphi$  to  $B$  is the same as the restriction of  $f$  to  $B$ . If  $A$  is a complete set of coset representatives of  $B$  in  $S$ , then each element  $x \in G$  is uniquely of the form  $x = a + u$  where  $a \in A$  and  $u \in G^1$ , and the mapping  $g$  defined by  $g(x) = \varphi(a) + f(u)$  is the desired isomorphism of  $G$  onto  $H$  which extends  $f$ .

The theorem now follows directly from (d).

#### REFERENCES

1. P. Crawley, *The cancellation of torsion abelian groups in direct sums*, J. Algebra **2** (1965), 432-442.
2. L. Fuchs, *Abelian groups*, Budapest, 1958.

3. J. Irwin and F. Richman, *Direct sums of countable groups and related concepts*, J. Algebra **2** (1965), 443-450.
4. I. Kaplansky, *Infinite abelian groups*, Ann Arbor, 1954.
5. G. Kolettis, *Direct sums of countable groups*, Duke Math. J. **27** (1960), 111-125.
6. L. Zippin, *Countable torsion groups*, Ann. of Math. **36** (1935), 86-99.

Received September 29, 1966. This work was supported in part by NSF Grant GP-3909.

CALIFORNIA INSTITUTE OF TECHNOLOGY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. SAMELSON  
Stanford University  
Stanford, California

J. P. JANS  
University of Washington  
Seattle, Washington 98105

J. DUGUNDJI  
University of Southern California  
Los Angeles, California 90007

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



# Pacific Journal of Mathematics

Vol. 22, No. 2

February, 1967

Paul Frank Baum, <i>Local isomorphism of compact connected Lie groups</i> . . . .	197
Lowell Wayne Beineke, Frank Harary and Michael David Plummer, <i>On the critical lines of a graph</i> . . . . .	205
Larry Eugene Bobisud, <i>On the behavior of the solution of the telegraphist's equation for large velocities</i> . . . . .	213
Richard Thomas Bumby, <i>Irreducible integers in Galois extensions</i> . . . . .	221
Chong-Yun Chao, <i>A nonimbedding theorem of nilpotent Lie algebras</i> . . . . .	231
Peter Crawley, <i>Abelian <math>p</math>-groups determined by their Ulm sequences</i> . . . . .	235
Bernard Russel Gelbaum, <i>Tensor products of group algebras</i> . . . . .	241
Newton Seymour Hawley, <i>Weierstrass points of plane domains</i> . . . . .	251
Paul Daniel Hill, <i>On quasi-isomorphic invariants of primary groups</i> . . . . .	257
Melvyn Klein, <i>Estimates for the transfinite diameter with applications to conformal mapping</i> . . . . .	267
Frederick M. Lister, <i>Simplifying intersections of disks in Bing's side approximation theorem</i> . . . . .	281
Charles Wisson McArthur, <i>On a theorem of Orlicz and Pettis</i> . . . . .	297
Harry Wright McLaughlin and Frederic Thomas Metcalf, <i>An inequality for generalized means</i> . . . . .	303
Daniel Russell McMillan, Jr., <i>Some topological properties of piercing points</i> . . . . .	313
Peter Don Morris and Daniel Eliot Wulbert, <i>Functional representation of topological algebras</i> . . . . .	323
Roger Wolcott Richardson, Jr., <i>On the rigidity of semi-direct products of Lie algebras</i> . . . . .	339
Jack Segal and Edward Sandusky Thomas, Jr., <i>Isomorphic cone-complexes</i> . . . . .	345
Richard R. Tucker, <i>The <math>\delta^2</math>-process and related topics</i> . . . . .	349
David Vere-Jones, <i>Ergodic properties of nonnegative matrices. I</i> . . . . .	361