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Let G, H, K be locally compact abelian groups where K is noncompact and both the quotient G/N^G where N^G is a compact (normal) subgroup and the quotient H/N^H where N^H is a compact (normal) subgroup. Then in a natural fashion the group algebras $L_1(G)$ and $L_1(H)$ are modules over $L_1(K)$ and

$$L_1(G) \bigotimes_{L_1(K)} L_1(H) \cong L_1(K) .$$

In [2, 3, 4, 5] there are discussions of tensor products of Banach spaces and Banach algebras over the field $\mathfrak{C}$ of complex numbers and over general Banach algebras. We note the following results to be found in these papers:

(i) If A, B, C are commutative Banach algebras and if A and B are bimodules over C (where $\| ca \| \leqq \| c \| \| a \|$, $\| cb \| \leqq \| c \| \| b \|$, $a \in A$, $b \in B$, $c \in C$) then the space $\mathfrak{M}_D$ of maximal ideals of $D \equiv A \otimes_C B$ may be identified with a subset of $\mathfrak{M}_A \times \mathfrak{M}_B$ as follows:

$$\mathfrak{M}_D = \{(M_A, M_B) : M_A \in \mathfrak{M}_A, M_B \in \mathfrak{M}_B, \mu(M_A) = \nu(M_B) \neq \text{null map}\} .$$

(Here μ and ν are continuous mappings of $\mathfrak{M}_A$ and $\mathfrak{M}_B$ into $\mathfrak{M}_C^\circ$ = the maximal ideal space of C with the null map adjoined. These maps are defined as follows: If $a \in A, b \in B, c \in C$ then

$$a^\wedge(M_A)c^\wedge(\mu(M_A)) = ca^\wedge(M_A)$$
$$b^\wedge(M_B)c^\wedge(\nu(M_B)) = cb^\wedge(M_B) .$$

Finally

$$c(a \otimes b)^\wedge(M_A, M_B) = c^\wedge(\mu(M_A))a^\wedge(M_A)b^\wedge(M_B)$$
$$= c^\wedge(\nu(M_B))a^\wedge(M_A)b^\wedge(M_B) .$$

[3].)

(ii) If G, H, K are locally compact abelian groups and if $\theta_G: K \to G$, $\theta_H: K \to H$ are continuous homomorphisms with closed images, then $L_1(G)$ and $L_1(H)$ are $L_1(K)$-bimodules according to the formulas:

$$ca(\xi) = \int_K a(\xi - \theta_G(\zeta))c(\zeta)d\zeta,\ a \in L_1(G),\ c \in L_1(K) .$$

$$cb(\eta) = \int_K b(\eta - \theta_H(\zeta))c(\zeta)d\zeta,\ b \in L_1(H),\ c \in L_1(K) .$$

Furthermore the mappings μ and ν of (i) are simply the dual mappings

$$\theta_G^\wedge: G^\wedge \to K^\wedge$$
$$\theta_H^\wedge: H^\wedge \to K^\wedge$$

of the character groups in question, [3, 4]. Finally,

$$L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(\mathfrak{G})$$

where

$$\mathfrak{G} = G \times H/(\theta_G \times \tilde{\theta}_H) \text{ diagonal } (K \times K) \text{ and } \tilde{\theta}_H(\zeta) = \theta_H(-\zeta) .$$

Loosely phrased, this says that the tensor product of group algebras is the group algebra of the tensor product of the groups.

The above results lead to the study of a similar (somewhat dual) situation described as follows:

Let G, H, K be locally compact abelian groups and let $\theta^G: G \to K$, $\theta^H: H \to K$ be continuous open homomorphisms with closed images. In what circumstances can $L_1(G)$ and $L_1(H)$ be made $L_1(K)$-bimodules relative to the mappings θ^G and θ^H? When these circumstances obtain, what is $\mathfrak{M}_D$, where $D = L_1(G) \otimes_{L_1(K)} L_1(H)$? Is there a group $\mathfrak{G}$ such that $D = L_1(\mathfrak{G})$?

We shall give answers to these questions in the following sections.

2. Examples. (i) Let G and K be compact abelian groups and let $\theta^G: G \to K$ be epic. Then define $L_1(G)$ as an $L_1(K)$-bimodule by:

$$ca(\xi) = \int_G a(\xi - \xi_1)\tilde{c}(\xi_1)d\xi_1$$

where $a \in L_1(G), c \in L_1(K)$ and $\tilde{c}(\xi) = c(\theta^G(\xi)), \tilde{c}(\eta) = c(\theta^H(\eta))$. (The above is defined first for continuous functions and then for arbitrary integrable functions by standard extension techniques.) Then

$$\| ca \| = \| \tilde{c} * a \| \leqq \| \tilde{c} \| \| a \| .$$

However, the map $F: c \to \int_G \tilde{c}(\xi_1)d\xi_1$ is a translation-invariant integral on $L_1(K)$. Thus we may and do assume

$$\int_G \tilde{c}(\xi_1)d\xi_1 = \int_K c(\zeta)d\zeta$$

and we conclude: $\| ca \| \leqq \| c \| \| a \|$.

(ii) Let $G = K = \mathfrak{R}$ = the set of real numbers. Let $\theta^G(\xi) = 2\xi$. Then for $c \in L_1(K)$ and $a \in L_1(G)$ let

$$ca(\xi) = \int_{-\infty}^{+\infty} a(\xi - \xi_1)c(2\xi_1)d\xi_1 .$$

In this case $\| ca \| \leqq \frac{1}{2}\| c \| \| a \|$.

(iii) If θ^G is not epic $F: L_1(K) \to \mathfrak{G}$ as defined in (i) need not be an invariant integral. For example, if $G = \{0\}$ and if K is an arbitrary nontrivial compact abelian group, then, for c continuous,

$$F(c) = \int_G \tilde{c}(\xi)d\xi = c(0) .$$

If $\zeta_0 \in K$ and if $c_0(\zeta) = c(\zeta + \zeta_0)$, then

$$F(c_0) = c_0(0) = c(\zeta_0) .$$

Thus, choosing c continuous and such that $c(0) \neq c(\zeta_0)$ we find F is not translation-invariant.

(iv) If G is *not* compact, if K is compact and even if θ^G is epic, then the action of $L_1(K)$ on $L_1(G)$ is not definable in the manner considered. Indeed, if $c(\zeta) \equiv 1$, and if $a \in L_1(G)$ we see

$$\begin{aligned} ca(\xi) &= \int_G a(\xi - \xi_1)\tilde{c}(\xi_1)d\xi_1 \\ &= \int_G a(\xi)d\xi , \end{aligned}$$

since $\tilde{c}(\xi_1) = c(\theta^G(\xi_1)) \equiv 1$. If, as we may, we choose a so that

$$\int_G a(\xi)d\xi \neq 0 ,$$

then $ca \notin L_1(G)$.

REMARK. Even if both G and K are not compact but if F is an invariant integral, the kernel of θ^G is compact. To prove this we assume, as we may, that Haar measures are adjusted so that

$$\int_K c(\zeta)d\zeta = \int_G \tilde{c}(\xi)d\xi = \int_H \tilde{c}(\eta)d\eta .$$

Furthermore, we may assume Haar measures on K and on ker $(\theta^G) \equiv N^G$ have been adjusted so that for $a \in L_1(G)$

$$\int_G a(\xi)d\xi = \int_K\left(\int_{N^G} a(\xi + \rho)d\rho\right)d\zeta ,$$

where ζ is the variable of integration on $K = G/N^G$. Since

$$\int_{N^G} a(\xi + \rho)d\rho$$

is constant on cosets of N^G, it may be regarded as a function of ζ. Then we find for any nontrivial nonnegative c in $L_1(K)$:

$$\int_G \tilde{c}(\xi)d\xi = \int_K \left(\int_{N^G} c(\theta^G(\xi + \rho))d\rho\right)d\zeta$$
$$= \int_K c(\zeta)d\zeta \cdot \int_{N^G} 1 d\rho$$

since $\rho \in \ker \theta^G$. Hence N^G must be compact, since otherwise

$$\int_{N^G} 1 d\rho = +\infty = \int_G \tilde{c}(\xi)d\xi = \int_K c(\zeta)d\zeta ,$$

a contradiction.

3. **The main formula.** In view of the conclusions of the preceding section, we posit the following situation:
(i) G, H, K are locally compact abelian groups.
(ii) $\theta^G: G \to K$, $\theta^H: H \to K$ are continuous open epimorphisms.
(iii) $L_1(G)$ and $L_1(H)$ are bimodules over $L_1(K)$ according to the actions:

$$ca(\xi) = \tilde{c} * a$$
$$cb(\eta) = \tilde{c} * b$$

where $a \in L_1(G)$, $b \in (H)$ and $c \in L_1(K)$. (Recall that

$$\tilde{c}(\xi) = c(\theta^G(\xi)),\ \tilde{c}(\eta) = c(\theta^H(\eta)) .)$$

(iv) Haar measures are adjusted so that the functionals

$$F_G: c \to \int_G c(\theta^G(\xi))d\xi = \int_G \tilde{c}(\xi)d\xi ,$$
$$F_H: c \to \int_H c(\theta^H(\eta))d\eta = \int_H \tilde{c}(\eta)d\eta$$

are translation-invariant integrals.

The argument used in the remark following (iv) of §2 shows:

If F is an invariant integral then

$$\int_G |\tilde{c}(\xi)|d\xi + \int_H |\tilde{c}(\eta)|d\eta < +\infty$$

if and only if N^G and N^H are compact.

In effect, we assume G, H, K are locally compact abelian groups and K is a noncompact quotient of both G and H by compact (normal) subgroups N^G and N^H.

Thus there is a wealth of concrete examples of the type that concerns us, e.g., $G = K \times N^G$, $H = K \times N^H$ where N^G and N^H are compact, K is locally compact and not compact and all groups are abelian.

In these circumstances

$$D \equiv L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(K) .$$

The formula is the conclusion of a sequence of lemmas. We recall that an interpretation of the results quoted in §1 may be given as follows:

(a)
$$\mathfrak{M}_{L_1(G)} = G^\wedge$$
$$\mathfrak{M}_{L_1(H)} = H^\wedge$$
$$\mathfrak{M}_{L_1(K)} = K^\wedge .$$

(b) There are mappings

$$\mu: G^\wedge \to K^\wedge \cup \{\text{null map}\}$$
$$\nu: H^\wedge \to K^\wedge \cup \{\text{null map}\}$$

and

$$\mathfrak{M}_D = \{(\alpha, \beta) : \alpha \in G^\wedge, \beta \in H^\wedge, \mu(\alpha) = \nu(\beta) \neq \text{null map}\} .$$

Furthermore

$$ca^\wedge(\alpha) = a^\wedge(\alpha)c^\wedge(\mu(\alpha)), a \in L_1(G), c \in L_1(K) ,$$
$$cb^\wedge(\beta) = b^\wedge(\beta)c^\wedge(\nu(\beta)), b \in L_1(H), c \in L_1(K) ,$$
$$\tilde{c}^\wedge(\alpha) = c^\wedge(\mu(\alpha)), \tilde{c}^\wedge(\beta) = c^\wedge(\nu(\beta)) .$$

Although we need never consider a pair (α, β) such that $\mu(\alpha) = \nu(\beta) =$ the null map sending $L_1(K)$ into 0, we shall have occasion to consider $\mu(\alpha)$ for all α and $\nu(\beta)$ for all β. Thus we shall interpret $c^\wedge(\mu(\alpha))$ and $c^\wedge(\nu(\beta))$ to be 0 if $\mu(\alpha) = \nu(\beta) =$ the null map, even though, since $c^\wedge$ is a function on $K^\wedge$, "$c^\wedge$(null map)" is not defined.

LEMMA 3.1. *The map* $L_1(K) \ni c(\zeta) \to \tilde{c}(\xi) \equiv c(\theta^G(\xi)) \in L_1(G)$ *is an isometric monomorphism. The image* $L_1(K)^G$ *of this map is a closed ideal in* $L_1(G)$. *Finally,* μ^{-1} *(null map)* $= h(L_1(K)^G) \equiv$ *hull* $(L_1(K)^G)$.

Proof. The algebraic and metric properties of the mapping are clear. To show $L_1(K)^G$ is an ideal (as the image of a complete space under an isometry $L_1(K)^G$ is closed) we consider c in $L_1(K)$ and a in $L_1(G)$. Then

$$a * \tilde{c} = \int_G a(\xi - \xi_1)c(\theta^G(\xi_1))d\xi_1$$
$$= \int_G a(\xi_2)c(\theta^G(\xi - \xi_2))d\xi_2 .$$

If $c_1(\zeta) = \int_G a(\xi_2)c(\zeta - \theta^G(\xi_2))d\xi_2$, then c_1 is in $L_1(K)$ and $\tilde{c}_1 = a * \tilde{c}$. Finally, if $\mu(\alpha) =$ (null map), then $c^\wedge(\mu(\alpha)) \equiv 0$ for all c in $L_1(K)$.

However, for a in $L_1(K)$ and such that $a^\wedge(\alpha) \neq 0$,

$$ca^\wedge(\alpha) = a^\wedge(\alpha)c^\wedge(\mu(\alpha)) = a^\wedge(\alpha)\int_G \tilde{c}(\xi)\overline{(\xi, \alpha)}d\xi$$

or

$$0 = \tilde{c}^\wedge(\mu(\alpha)) = \tilde{c}^\wedge(\alpha) .$$

Thus $\alpha \in h(L_1(K)^G)$, i.e., μ^{-1} (null map) $\subset h(L_1(K)^G)$.

Conversely, if $\alpha \in h(L_1(K)^G)$, then $\tilde{c}^\wedge(\alpha) \equiv 0$ for all c in $L_1(K)$. The above formulas show $c^\wedge(\mu(\alpha)) \equiv 0$ for all c in $L_1(K)$, whence $\mu(\alpha) =$ (null map) and we conclude μ^{-1}(null map) $= h(L_1(K)^G)$.

Let $\hat{\theta}^G, \hat{\theta}^H$ be the duals of the maps θ^G, θ^H. Thus, e.g., $(\xi, \hat{\theta}^G(\gamma)) = (\theta^G(\xi), \gamma)$ for all $\gamma \in \hat{K}$. If S is a set in G, let $S^\perp$ be the "annihilator" of S, i.e., the set of α in $\hat{G}$ such that $(s, \alpha) = 1$ for all $s \in S$. We prove

LEMMA 3.2. (a) $N^{G\perp} = \hat{\theta}^G\hat{K}$;

(b) $\hat{G} = N^{G\perp} \cup h(L_1(K)^G), \emptyset = N^{G\perp} \cap h(L_1(K)^G)$;

(c) $\mu: N^{G\perp} \to \hat{K}$ *is an isomophism* [6, p. 103].

Proof. (a) If $\xi \in N^G$ then $\theta^G(\xi) =$ identity and $(\theta^G(\xi), \gamma) = 1$ for all $\gamma \in \hat{K}$. Thus $\hat{\theta}^G(\hat{K}) \subset N^{G\perp}$. If $\alpha \in N^{G\perp}$, then for all $\xi \in N^G$, $(\xi, \alpha) = 1$. If $\alpha \notin \hat{\theta}^G(\hat{K})$, then, since $\hat{\theta}^G(\hat{K})$ is closed, there is a ξ_0 such that

$$(\xi_0, \alpha) \neq 1, (\xi_0, \hat{\theta}^G(\hat{K})) = 1 = (\theta^G(\xi_0), \hat{K}), \text{ i.e., } \xi_0 \in N^G ,$$

a contradiction. Thus $\hat{\theta}^G(\hat{K}) = N^{G\perp}$, $\mu(N^{G\perp}) = \mu(\hat{\theta}^G(\hat{K})) = \hat{K}$.

(b) and (c) If $\alpha_0 \notin N^{G\perp}$ then $\mu(\alpha_0) =$ (null map). For if $\alpha_0 \notin N^{G\perp}$, then α_0 may be regarded as a nontrivial character of the compact group N^G. Thus $\int_{N^G}(\xi + \rho, \alpha_0)d\rho = \int_{N^G}(\xi, \alpha_0)(\rho, \alpha_0)d\rho = 0$. Hence if $c \in L_1(K)$ then

$$\begin{aligned} c^\wedge(\mu(\alpha_0)) &= \int_G c(\theta^G(\xi))\overline{(\xi, \alpha_0)}d\xi \\ &= \int_K\left(\int_{N^G} c(\theta^G(\xi + \rho))(\xi + \rho, \alpha_0)d\rho\right)d\xi \\ &= \int_K c(\zeta)\left(\int_{N^G}(\xi + \rho, \alpha_0)d\rho\right)d\zeta = 0 . \end{aligned}$$

Thus $\mu(\alpha_0) =$ (null map), and $\hat{G}\backslash N^{G\perp} \subset h(L_1(K)^G)$. On the other hand if α is in $h(L_1(K)^G)$ then α is not in $N^{G\perp}$. Otherwise, α may be viewed as some γ in $\hat{K}$ and thus for c in $L_1(K)$ we have

$$\begin{aligned}\tilde{c}^{\wedge}(\alpha) = 0 &= \int_G c(\theta^G(\xi))\overline{(\xi, \alpha)}d\xi \\ &= \int_K \Big(\int_{N^G} c(\theta^G(\xi + \rho))\overline{(\xi + \rho, \alpha)}d\rho\Big)d\zeta \\ &= \int_K c(\zeta)\overline{(\zeta, \gamma)}d\zeta \int_{N^G} 1 d\rho \ .\end{aligned}$$

Hence $c^{\wedge}(\gamma) = 0$ for all c in $L_1(K)$, a contradiction. Thus $\hat{G}/N^{G\perp} = h(L_1(K)^G)$ and we conclude the truth of (b).

Next, if $\hat{\theta}^G(\gamma) = \alpha$ then for c in $L_1(K)$ and a in $L_1(G)$

$$\begin{aligned}ca^{\wedge}(\alpha) &= a^{\wedge}(\alpha)\int_G c(\theta^G(\xi))\overline{(\xi, \hat{\theta}^G(\gamma))}d\xi \\ &= a^{\wedge}(\alpha)c^{\wedge}(\gamma) \ .\end{aligned}$$

Hence, $c^{\wedge}(\mu(\alpha)) = c^{\wedge}(\gamma)$ and $\mu(\alpha) = \gamma = \mu\hat{\theta}^G(\gamma)$.

Clearly

$$\begin{aligned}\mu(\hat{\theta}^G(\gamma_1)\hat{\theta}^G(\gamma_2)) &= \mu(\hat{\theta}^G(\gamma_1\gamma_2)) = \gamma_1\gamma_2 \\ &= \mu\hat{\theta}^G(\gamma_1)\mu\hat{\theta}^G(\gamma_2) \ .\end{aligned}$$

Thus μ is an epimorphism of $\hat{\theta}^G(K)^{\wedge}$ onto $K^{\wedge}$ and $\mu\hat{\theta}^G$ is the identity. It follows that μ is one-to-one on $\hat{\theta}^G(K)$ and furthermore that $\hat{\theta}^G\mu$ is the identity on $\hat{\theta}^G K : \hat{\theta}^G\mu(\hat{\theta}^G(\gamma)) = \hat{\theta}^G(\gamma)$.

Combining our results to this point we see that

$$\mathfrak{M}_D = \mathrm{diag}\,(K^{\wedge} \times K^{\wedge}) \cong K^{\wedge} \ .$$

It follows that K is a reasonable candidate for the group $\mathfrak{G}$ such that $D \cong L_1(\mathfrak{G})$. Indeed, if $\mathfrak{G}$ is such a group then $\mathfrak{G}^{\wedge} = \mathfrak{M}_D$. Since $\mathfrak{M}_D = K^{\wedge}$, we conclude $\mathfrak{G} = K$.

We shall now define a map $T: D \rightarrow L_1(K)$. As usual T is defined on

$$\begin{aligned}\mathfrak{F} &\equiv F_{L_1(K)}(L_1(G), L_1(H)) \\ &= \Big\{f : f : L_1(G) \times L_1(H) \rightarrow L_1(K), \|f\| \\ &\equiv \sum_{(a,b)} \|f(a, b)\| \, \|a\| \, \|b\| < \infty, f(0, b) = f(a, 0) + 0\Big\}\end{aligned}$$

[2, 3]. Thus if $c(a, b)$ is the function taking the value c at (a, b) we set

$$T(c(a, b)) = \int_{N^G} ca(\xi + \rho)d\rho * \int_{N^H} b(\eta + \sigma)d\sigma$$

where $N^H = \ker(\theta^H)$. We note that each of the integrals above is a function on K and hence so is the indicated convolution. It is a simple matter to verify that when T is extended by linearity it is a bounded epimorphism of the algebra $\mathfrak{F}$ onto $L_1(K)$ and that T annihilates the reducing ideal I, modulo which the algebra $\mathfrak{F}$ is D. (The surjectivity of T follows from the fact that the integrals $\int_{N^G} \equiv T_G$ and $\int_{N^H} \equiv T_H$ are epimorphisms, from a simple application of approximate identities and from P. J. Cohen's factorization theorem [1, 3, 4].)

We show now for T, which may be regarded as a mapping of D onto $L_1(K)$,

LEMMA 3.3. *T is an isomorphism if and only if D is semisimple.*

Proof. Clearly, if T is an isomorphism then D is semisimple.

Conversely, if D is semisimple and if $T(z) = 0$, where $z = \sum_{n=1}^{\infty} c_n(a_n \otimes b_n)$ [2, 3], then for any γ in $K^\wedge$, $T^\wedge(z)(\gamma) = 0$. Thus

$$\begin{aligned} T^\wedge(z)(\gamma) &= \sum_{n=1}^{\infty} \widehat{T_G(c_n a_n)}(\gamma)\widehat{T_H(b_n)}(\gamma) \\ &= \sum_{n=1}^{\infty} \hat{c_n}(\gamma)\widehat{T_G(a_n)}(\gamma)\widehat{T_H(b_n)}(\gamma) = 0 . \end{aligned}$$

However,

$$\begin{aligned} \hat{T_G}(a)(\gamma) &= \int_K T_G(a)(\zeta)\overline{(\zeta, \gamma)}d\zeta \\ &= \int_K\left(\int_{N^G} a(\xi + \rho)d\rho\right)\overline{(\zeta, \gamma)}d\zeta \\ &= \int_K\left(\int_{N^G} a(\xi + \rho)\overline{(\xi + \rho, \gamma)}d\rho\right)d\zeta \\ &= a^\wedge(\alpha) \end{aligned}$$

where $\alpha = \hat{\theta}^G(\gamma)$. After similar arguments about T_H we find

$$T^\wedge(z)(\gamma) = \sum_{n=1}^{\infty} \hat{c_n}(\gamma)\hat{a_n}(\alpha)\hat{b_n}(\beta)$$

where $\beta = \hat{\theta}^H(\gamma)$. In other words $T^\wedge(z)(\gamma) = z^\wedge(\alpha, \beta)$ where $\mu(\alpha) = \gamma(\beta)$ and (α, β) corresponds to an element of $\mathfrak{M}_D$. Since $T^\wedge(z)(\gamma) \equiv 0$ for all γ, we find $z^\wedge(\alpha, \beta) \equiv 0$ for all (α, β) corresponding to elements of $\mathfrak{M}_D$. The semisimplicity assumption now shows $z = 0$ and hence that T is an isomorphism.

We now conclude by proving

LEMMA 3.4. *D is semisimple.*

Proof. Let z belong to the radical of D. As in [3, 4] we may assume that z is of the form $\sum_{n=1}^{\infty} c_n(a_n \otimes b_n)$ where, for fixed compact sets U, V, W in $G^\wedge$, $H^\wedge$, $K^\wedge$ and for all n, support $\hat{a_n}(\alpha) \subset U$, support $\hat{b_n}(\beta) \subset V$, and support $\hat{c_n}(\gamma) \subset W$. Furthermore, we may assume that each c_n is of the form $c_{n1} * c_{n2} * c_{n3}$ and thus in effect that

$$z = \sum_{n=1}^{\infty} c_{n1}(c_{n2}a_n \otimes c_{n3}b_n)$$

where support $\hat{c_{n1}}(\gamma) \subset W$.

Since $L_1(K)^G$ is an ideal in $L_1(G)$ and since there is a corresponding statement for $L_1(K)^H$, we conclude that there are elements d_{n2}, d_{n3} in $L_1(K)$ such that $\tilde{d}_{n2}(\xi) = c_{n2}a_n(\xi)$, $\tilde{d}_{n3}(\eta) = c_{n3}b_n(\eta)$.

Furthermore, $\hat{\tilde{d}_{n2}}(\alpha) = d_{n2}(\mu(\alpha))$, $\hat{\tilde{d}_{n3}}(\beta) = d_{n3}(\nu(\beta))$, and $\hat{d_{n2}}(\gamma) \neq 0$, or $\hat{d_{n3}}(\gamma) \neq 0$ implies $\hat{d_{n2}}(\mu\hat{\theta}^G(\gamma)) = \hat{\tilde{d}_{n2}}(\hat{\theta}^G(\gamma)) \neq 0$, etc., i.e., that $\gamma \in \mu$ (support $\hat{\tilde{d}_{n2}}$), etc. Thus there is a fixed compact set Y containing the supports of all $\hat{c_{n1}}$, $\hat{c_{n2}}$, $\hat{c_{n3}}$, $\hat{d_{n1}}$, $\hat{d_{n2}}$, $\hat{d_{n3}}$. Hence there is a fixed c in $L_1(K)$ such that $c^\wedge(\gamma) \equiv 1$ on Y, support $c^\wedge(\gamma)$ is compact and

$$0 \leqq c^\wedge(\gamma) \leqq 1 .$$

For this c it is true that $c_{nj} = c_{nj} * c$, $d_{nj} = d_{nj} * c$, $j = 1, 2, 3$. Thus we find

$$\begin{aligned} z &= \sum_{n=1}^{\infty} c_n(a_n \otimes b_n) = \sum_{n=1}^{\infty} c_{n1}(c_{n2}a_n \otimes c_{n3}b_n) \\ &= \sum_{n=1}^{\infty} c_{n1}(d_{n2} \otimes d_{n3}) = \sum_{n=1}^{\infty} c_{n1}(d_{n2}c \otimes d_{n3}c) \\ &= \left(\sum_{n=1}^{\infty} c_{n1}d_{n2}d_{n3}\right)(c \otimes c) . \end{aligned}$$

However, for all γ in $K^\wedge$

$$\widehat{c_{n1}d_{n2}d_{n3}}(\gamma) = \hat{c_{n1}}(\gamma)\hat{d_{n2}}(\gamma)\hat{d_{n3}}(\gamma) .$$

Furthermore

$$\begin{aligned} d_{n2}(\zeta) &= \int_G a_n(\xi)c_{n2}(\zeta - \theta^G(\xi))d\xi \\ d_{n3}(\zeta) &= \int_H b_n(\eta)c_{n3}(\zeta - \theta^H(\eta))d\eta . \end{aligned}$$

Thus

$$\begin{aligned} \hat{d_{n2}}(\gamma) &= \int_G\int_K a_n(\xi)c_{n2}(\zeta - \theta^G(\xi))\overline{(\zeta, \gamma)}d\xi d\zeta \\ &= \int_G\int_K a_n(\xi)c_{n2}(\zeta_1)\overline{(\zeta_1, \gamma)(\theta^G(\xi), \gamma)}d\xi_1 d\zeta \\ &= \hat{a_n}(\hat{\theta}^G(\gamma))c_{n2}(\gamma) \end{aligned}$$

and similarly $\hat{d_{n3}}(\gamma) = b_n(\hat{\theta}^H(\gamma))c_{n3}(\gamma)$. We see then that

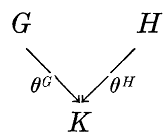
$$\widehat{c_{n1}}(\gamma)\widehat{d_{n2}}(\gamma)\widehat{d_{n3}}(\gamma) = \widehat{c_{n1}}(\gamma)\widehat{c_{n2}}(\gamma)\widehat{a_n}(\hat{\theta}^G(\gamma))\widehat{c_{n3}}(\gamma)\widehat{b_n}(\hat{\theta}^H(\gamma))$$

and since $\mu\hat{\theta}^G(\gamma) = \nu\hat{\theta}^H(\gamma) = \gamma$ we conclude that

$$\sum_{n=1}^{\infty} \widehat{c_{n1}}(\gamma)\widehat{d_{n2}}(\gamma)\widehat{d_{n3}}(\gamma) = z^{\wedge}(\{\hat{\theta}^G(\gamma),\, \hat{\theta}^H(\gamma)\})$$

which is zero as a consequence of our assumption. Thus $z = 0$ and the semisimplicity of D is established.

Hence, in the context indicated above and suggested by the diagram



there obtains the formula

$$L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(K) .$$

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