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TENSOR PRODUCTS OF GROUP ALGEBRAS

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Let G, H, K be locally compact abelian groups where K is noncompact and both the quotient G/N^G where N^G is a compact (normal) subgroup and the quotient H/N^H where N^H is a compact (normal) subgroup. Then in a natural fashion the group algebras $L_1(G)$ and $L_1(H)$ are modules over $L_1(K)$ and

$$L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(K).$$

In [2, 3, 4, 5] there are discussions of tensor products of Banach spaces and Banach algebras over the field \mathbb{C} of complex numbers and over general Banach algebras. We note the following results to be found in these papers:

(i) If A, B, C are commutative Banach algebras and if A and B are bimodules over C (where $\|ca\| \leq \|c\| \|a\|$, $\|cb\| \leq \|c\| \|b\|$, $a \in A$, $b \in B$, $c \in C$) then the space \mathfrak{M}_D of maximal ideals of $D \equiv A \otimes_C B$ may be identified with a subset of $\mathfrak{M}_A \times \mathfrak{M}_B$ as follows:

$$\mathfrak{M}_D = \{(M_A, M_B) : M_A \in \mathfrak{M}_A, M_B \in \mathfrak{M}_B, \mu(M_A) = \nu(M_B) \neq \text{null map}\}.$$

(Here μ and ν are continuous mappings of \mathfrak{M}_A and \mathfrak{M}_B into $\mathfrak{M}_C^0 =$ the maximal ideal space of C with the null map adjoined. These maps are defined as follows: If $a \in A, b \in B, c \in C$ then

$$\begin{aligned} a^\wedge(M_A)c^\wedge(\mu(M_A)) &= ca^\wedge(M_A) \\ b^\wedge(M_B)c^\wedge(\nu(M_B)) &= cb^\wedge(M_B). \end{aligned}$$

Finally

$$\begin{aligned} c(a \otimes b)^\wedge(M_A, M_B) &= c^\wedge(\mu(M_A))a^\wedge(M_A)b^\wedge(M_B) \\ &= c^\wedge(\nu(M_B))a^\wedge(M_A)b^\wedge(M_B). \end{aligned}$$

[3].)

(ii) If G, H, K are locally compact abelian groups and if $\theta_G: K \rightarrow G$, $\theta_H: K \rightarrow H$ are continuous homomorphisms with closed images, then $L_1(G)$ and $L_1(H)$ are $L_1(K)$ -bimodules according to the formulas:

$$\begin{aligned} ca(\xi) &= \int_K a(\xi - \theta_G(\zeta))c(\zeta)d\zeta, \quad a \in L_1(G), \quad c \in L_1(K). \\ cb(\eta) &= \int_K b(\eta - \theta_H(\zeta))c(\zeta)d\zeta, \quad b \in L_1(H), \quad c \in L_1(K). \end{aligned}$$

Furthermore the mappings μ and ν of (i) are simply the dual mappings

$$\begin{aligned} \theta_G^\wedge: G^\wedge &\rightarrow K^\wedge \\ \theta_H^\wedge: H^\wedge &\rightarrow K^\wedge \end{aligned}$$

of the character groups in question, [3, 4]. Finally,

$$L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(\mathbb{G})$$

where

$$\mathbb{G} = G \times H / (\theta_G \times \tilde{\theta}_H) \text{ diagonal } (K \times K) \text{ and } \tilde{\theta}_H(\zeta) = \theta_H(-\zeta).$$

Loosely phrased, this says that the tensor product of group algebras is the group algebra of the tensor product of the groups.

The above results lead to the study of a similar (somewhat dual) situation described as follows:

Let G, H, K be locally compact abelian groups and let $\theta^G: G \rightarrow K$, $\theta^H: H \rightarrow K$ be continuous open homomorphisms with closed images. In what circumstances can $L_1(G)$ and $L_1(H)$ be made $L_1(K)$ -bimodules relative to the mappings θ^G and θ^H ? When these circumstances obtain, what is \mathfrak{M}_D , where $D = L_1(G) \otimes_{L_1(K)} L_1(H)$? Is there a group \mathbb{G} such that $D = L_1(\mathbb{G})$?

We shall give answers to these questions in the following sections.

2. Examples. (i) Let G and K be compact abelian groups and let $\theta^G: G \rightarrow K$ be epic. Then define $L_1(G)$ as an $L_1(K)$ -bimodule by:

$$ca(\xi) = \int_a a(\xi - \xi_1) \tilde{c}(\xi_1) d\xi_1$$

where $a \in L_1(G)$, $c \in L_1(K)$ and $\tilde{c}(\xi) = c(\theta^G(\xi))$, $\tilde{c}(\gamma) = c(\theta^H(\gamma))$. (The above is defined first for continuous functions and then for arbitrary integrable functions by standard extension techniques.) Then

$$\|ca\| = \|\tilde{c} * a\| \leq \|\tilde{c}\| \|a\|.$$

However, the map $F: c \rightarrow \int_a \tilde{c}(\xi_1) d\xi_1$ is a translation-invariant integral on $L_1(K)$. Thus we may and do assume

$$\int_a \tilde{c}(\xi_1) d\xi_1 = \int_K c(\zeta) d\zeta$$

and we conclude: $\|ca\| \leq \|c\| \|a\|$.

(ii) Let $G = K = \mathfrak{R} =$ the set of real numbers. Let $\theta^G(\xi) = 2\xi$. Then for $c \in L_1(K)$ and $a \in L_1(G)$ let

$$ca(\xi) = \int_{-\infty}^{+\infty} a(\xi - \xi_1) c(2\xi_1) d\xi_1.$$

In this case $\|ca\| \leq \frac{1}{2} \|c\| \|a\|$.

(iii) If θ^a is not epic $F: L_1(K) \rightarrow \mathfrak{G}$ as defined in (i) need not be an invariant integral. For example, if $G = \{0\}$ and if K is an arbitrary nontrivial compact abelian group, then, for c continuous,

$$F(c) = \int_a \tilde{c}(\xi) d\xi = c(0) .$$

If $\zeta_0 \in K$ and if $c_0(\zeta) = c(\zeta + \zeta_0)$, then

$$F(c_0) = c_0(0) = c(\zeta_0) .$$

Thus, choosing c continuous and such that $c(0) \neq c(\zeta_0)$ we find F is not translation-invariant.

(iv) If G is *not* compact, if K is compact and even if θ^a is epic, then the action of $L_1(K)$ on $L_1(G)$ is not definable in the manner considered. Indeed, if $c(\zeta) \equiv 1$, and if $a \in L_1(G)$ we see

$$\begin{aligned} ca(\xi) &= \int_a a(\xi - \xi_1) \tilde{c}(\xi_1) d\xi_1 \\ &= \int_a a(\xi) d\xi , \end{aligned}$$

since $\tilde{c}(\xi_1) = c(\theta^a(\xi_1)) \equiv 1$. If, as we may, we choose a so that

$$\int_a a(\xi) d\xi \neq 0 ,$$

then $ca \notin L_1(G)$.

REMARK. Even if both G and K are not compact but if F is an invariant integral, the kernel of θ^a is compact. To prove this we assume, as we may, that Haar measures are adjusted so that

$$\int_K c(\zeta) d\zeta = \int_a \tilde{c}(\xi) d\xi = \int_H \tilde{c}(\eta) d\eta .$$

Furthermore, we may assume Haar measures on K and on $\ker(\theta^a) \equiv N^a$ have been adjusted so that for $a \in L_1(G)$

$$\int_a a(\xi) d\xi = \int_K \left(\int_{N^a} a(\xi + \rho) d\rho \right) d\zeta ,$$

where ζ is the variable of integration on $K = G/N^a$. Since

$$\int_{N^a} a(\xi + \rho) d\rho$$

is constant on cosets of N^a , it may be regarded as a function of ζ . Then we find for any nontrivial nonnegative c in $L_1(K)$:

$$\begin{aligned} \int_{\mathcal{G}} \tilde{c}(\xi) d\xi &= \int_{\mathcal{K}} \left(\int_{N^{\mathcal{G}}} c(\theta^{\mathcal{G}}(\xi + \rho)) d\rho \right) d\xi \\ &= \int_{\mathcal{K}} c(\zeta) d\zeta \cdot \int_{N^{\mathcal{G}}} 1 d\rho \end{aligned}$$

since $\rho \in \ker \theta^{\mathcal{G}}$. Hence $N^{\mathcal{G}}$ must be compact, since otherwise

$$\int_{N^{\mathcal{G}}} 1 d\rho = +\infty = \int_{\mathcal{G}} \tilde{c}(\xi) d\xi = \int_{\mathcal{K}} c(\zeta) d\zeta,$$

a contradiction.

3. The main formula. In view of the conclusions of the preceding section, we posit the following situation:

- (i) G, H, K are locally compact abelian groups.
- (ii) $\theta^{\mathcal{G}}: G \rightarrow K, \theta^{\mathcal{H}}: H \rightarrow K$ are continuous open epimorphisms.
- (iii) $L_1(G)$ and $L_1(H)$ are bimodules over $L_1(K)$ according to the actions:

$$\begin{aligned} ca(\xi) &= \tilde{c} * a \\ cb(\eta) &= \tilde{c} * b \end{aligned}$$

where $a \in L_1(G), b \in (H)$ and $c \in L_1(K)$. (Recall that

$$\tilde{c}(\xi) = c(\theta^{\mathcal{G}}(\xi)), \tilde{c}(\eta) = c(\theta^{\mathcal{H}}(\eta)) .)$$

- (iv) Haar measures are adjusted so that the functionals

$$\begin{aligned} F_{\mathcal{G}}: c &\rightarrow \int_{\mathcal{G}} c(\theta^{\mathcal{G}}(\xi)) d\xi = \int_{\mathcal{G}} \tilde{c}(\xi) d\xi, \\ F_{\mathcal{H}}: c &\rightarrow \int_{\mathcal{H}} c(\theta^{\mathcal{H}}(\eta)) d\eta = \int_{\mathcal{H}} \tilde{c}(\eta) d\eta \end{aligned}$$

are translation-invariant integrals.

The argument used in the remark following (iv) of §2 shows:

If F is an invariant integral then

$$\int_{\mathcal{G}} |\tilde{c}(\xi)| d\xi + \int_{\mathcal{H}} |\tilde{c}(\eta)| d\eta < +\infty$$

if and only if $N^{\mathcal{G}}$ and $N^{\mathcal{H}}$ are compact.

In effect, we assume G, H, K are locally compact abelian groups and K is a noncompact quotient of both G and H by compact (normal) subgroups $N^{\mathcal{G}}$ and $N^{\mathcal{H}}$.

Thus there is a wealth of concrete examples of the type that concerns us, e.g., $G = K \times N^{\mathcal{G}}, H = K \times N^{\mathcal{H}}$ where $N^{\mathcal{G}}$ and $N^{\mathcal{H}}$ are compact, K is locally compact and not compact and all groups are abelian.

In these circumstances

$$D \equiv L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(K) .$$

The formula is the conclusion of a sequence of lemmas. We recall that an interpretation of the results quoted in §1 may be given as follows:

- (a) $\mathfrak{M}_{L_1(G)} = G^\wedge$
 $\mathfrak{M}_{L_1(H)} = H^\wedge$
 $\mathfrak{M}_{L_1(K)} = K^\wedge .$

(b) There are mappings

$$\begin{aligned} \mu: G^\wedge &\rightarrow K^\wedge \cup \{\text{null map}\} \\ \nu: H^\wedge &\rightarrow K^\wedge \cup \{\text{null map}\} \end{aligned}$$

and

$$\mathfrak{M}_D = \{(\alpha, \beta) : \alpha \in G^\wedge, \beta \in H^\wedge, \mu(\alpha) = \nu(\beta) \neq \text{null map}\} .$$

Furthermore

$$\begin{aligned} c\hat{a}(\alpha) &= a^\wedge(\alpha)c^\wedge(\mu(\alpha)), a \in L_1(G), c \in L_1(K) , \\ c\hat{b}(\beta) &= b^\wedge(\beta)c^\wedge(\nu(\beta)), b \in L_1(H), c \in L_1(K) , \\ \tilde{c}^\wedge(\alpha) &= c^\wedge(\mu(\alpha)), \tilde{c}^\wedge(\beta) = c^\wedge(\nu(\beta)) . \end{aligned}$$

Although we need never consider a pair (α, β) such that $\mu(\alpha) = \nu(\beta) =$ the null map sending $L_1(K)$ into 0, we shall have occasion to consider $\mu(\alpha)$ for all α and $\nu(\beta)$ for all β . Thus we shall interpret $c^\wedge(\mu(\alpha))$ and $c^\wedge(\nu(\beta))$ to be 0 if $\mu(\alpha) = \nu(\beta) =$ the null map, even though, since c^\wedge is a function on K^\wedge , “ $c^\wedge(\text{null map})$ ” is not defined.

LEMMA 3.1. *The map $L_1(K) \ni c(\zeta) \rightarrow \tilde{c}(\xi) \equiv c(\theta^\alpha(\xi)) \in L_1(G)$ is an isometric monomorphism. The image $L_1(K)^\alpha$ of this map is a closed ideal in $L_1(G)$. Finally, $\mu^{-1}(\text{null map}) = h(L_1(K)^\alpha) \equiv \text{hull}(L_1(K)^\alpha)$.*

Proof. The algebraic and metric properties of the mapping are clear. To show $L_1(K)^\alpha$ is an ideal (as the image of a complete space under an isometry $L_1(K)^\alpha$ is closed) we consider c in $L_1(K)$ and a in $L_1(G)$. Then

$$\begin{aligned} a * \tilde{c} &= \int_{\mathcal{G}} a(\xi - \xi_1)c(\theta^\alpha(\xi_1))d\xi_1 \\ &= \int_{\mathcal{G}} a(\xi_2)c(\theta^\alpha(\xi - \xi_2))d\xi_2 . \end{aligned}$$

If $c_1(\zeta) = \int_{\mathcal{G}} a(\xi_2)c(\zeta - \theta^\alpha(\xi_2))d\xi_2$, then c_1 is in $L_1(K)$ and $\tilde{c}_1 = a * \tilde{c}$. Finally, if $\mu(\alpha) = (\text{null map})$, then $c^\wedge(\mu(\alpha)) \equiv 0$ for all c in $L_1(K)$.

However, for a in $L_1(K)$ and such that $a^\wedge(\alpha) \neq 0$,

$$c^\wedge(\alpha) = a^\wedge(\alpha)c^\wedge(\mu(\alpha)) = a^\wedge(\alpha) \int_G \tilde{c}(\xi) \overline{(\xi, \alpha)} d\xi$$

or

$$0 = \tilde{c}^\wedge(\mu(\alpha)) = \tilde{c}^\wedge(\alpha).$$

Thus $\alpha \in h(L_1(K)^G)$, i.e., $\mu^{-1}(\text{null map}) \subset h(L_1(K)^G)$.

Conversely, if $\alpha \in h(L_1(K)^G)$, then $\tilde{c}^\wedge(\alpha) \equiv 0$ for all c in $L_1(K)$. The above formulas show $c^\wedge(\mu(\alpha)) \equiv 0$ for all c in $L_1(K)$, whence $\mu(\alpha) = (\text{null map})$ and we conclude $\mu^{-1}(\text{null map}) = h(L_1(K)^G)$.

Let $\hat{\theta}^\alpha, \hat{\theta}^\beta$ be the duals of the maps $\theta^\alpha, \theta^\beta$. Thus, e.g., $(\xi, \hat{\theta}^\alpha(\gamma)) = (\theta^\alpha(\xi), \gamma)$ for all $\gamma \in \hat{K}$. If S is a set in G , let S^\perp be the "annihilator" of S , i.e., the set of α in \hat{G} such that $(s, \alpha) = 1$ for all $s \in S$. We prove

LEMMA 3.2. (a) $N^{G^\perp} = \hat{\theta}^\alpha \hat{K}$;

(b) $\hat{G} = N^{G^\perp} \cup h(L_1(K)^G), \emptyset = N^{G^\perp} \cap h(L_1(K)^G)$;

(c) $\mu: N^{G^\perp} \rightarrow \hat{K}$ is an isomorphism [6, p. 103].

Proof. (a) If $\xi \in N^G$ then $\theta^\alpha(\xi) = \text{identity}$ and $(\theta^\alpha(\xi), \gamma) = 1$ for all $\gamma \in \hat{K}$. Thus $\hat{\theta}^\alpha(\hat{K}) \subset N^{G^\perp}$. If $\alpha \in N^{G^\perp}$, then for all $\xi \in N^G$, $(\xi, \alpha) = 1$. If $\alpha \notin \hat{\theta}^\alpha(\hat{K})$, then, since $\hat{\theta}^\alpha(\hat{K})$ is closed, there is a ξ_0 such that

$$(\xi_0, \alpha) \neq 1, (\xi_0, \hat{\theta}^\alpha(\hat{K})) = 1 = (\theta^\alpha(\xi_0), \hat{K}), \text{ i.e., } \xi_0 \in N^G,$$

a contradiction. Thus $\hat{\theta}^\alpha(\hat{K}) = N^{G^\perp}$, $\mu(N^{G^\perp}) = \mu(\hat{\theta}^\alpha(\hat{K})) = \hat{K}$.

(b) and (c) If $\alpha_0 \notin N^{G^\perp}$ then $\mu(\alpha_0) = (\text{null map})$. For if $\alpha_0 \in N^{G^\perp}$, then α_0 may be regarded as a nontrivial character of the compact group N^G . Thus $\int_{N^G} (\xi + \rho, \alpha_0) d\rho = \int_{N^G} (\xi, \alpha_0)(\rho, \alpha_0) d\rho = 0$. Hence if $c \in L_1(K)$ then

$$\begin{aligned} c^\wedge(\mu(\alpha_0)) &= \int_G c(\theta^\alpha(\xi)) \overline{(\xi, \alpha_0)} d\xi \\ &= \int_K \left(\int_{N^G} c(\theta^\alpha(\xi + \rho)) (\xi + \rho, \alpha_0) d\rho \right) d\xi \\ &= \int_K c(\zeta) \left(\int_{N^G} (\xi + \rho, \alpha_0) d\rho \right) d\zeta = 0. \end{aligned}$$

Thus $\mu(\alpha_0) = (\text{null map})$, and $\hat{G} \setminus N^{G^\perp} \subset h(L_1(K)^G)$. On the other hand if α is in $h(L_1(K)^G)$ then α is not in N^{G^\perp} . Otherwise, α may be viewed as some γ in \hat{K} and thus for c in $L_1(K)$ we have

$$\begin{aligned} \widehat{c}^\wedge(\alpha) = 0 &= \int_{\mathfrak{g}} c(\theta^{\mathfrak{g}}(\xi)) \overline{(\xi, \alpha)} d\xi \\ &= \int_{\mathfrak{K}} \left(\int_{N^{\mathfrak{g}}} c(\theta^{\mathfrak{g}}(\xi + \rho)) \overline{(\xi + \rho, \alpha)} d\rho \right) d\xi \\ &= \int_{\mathfrak{K}} c(\zeta) \overline{(\zeta, \gamma)} d\zeta \int_{N^{\mathfrak{g}}} 1 d\rho . \end{aligned}$$

Hence $c^\wedge(\gamma) = 0$ for all c in $L_1(K)$, a contradiction. Thus $\widehat{G}/N^{G^\perp} = h(L_1(K)^{\mathfrak{g}})$ and we conclude the truth of (b).

Next, if $\widehat{\theta}^{\mathfrak{g}}(\gamma) = \alpha$ then for c in $L_1(K)$ and a in $L_1(G)$

$$\begin{aligned} ca^\wedge(\alpha) &= a^\wedge(\alpha) \int_{\mathfrak{g}} c(\theta^{\mathfrak{g}}(\xi)) \overline{(\xi, \widehat{\theta}^{\mathfrak{g}}(\gamma))} d\xi \\ &= a^\wedge(\alpha) c^\wedge(\gamma) . \end{aligned}$$

Hence, $c^\wedge(\mu(\alpha)) = c^\wedge(\gamma)$ and $\mu(\alpha) = \gamma = \mu\widehat{\theta}^{\mathfrak{g}}(\gamma)$.

Clearly

$$\begin{aligned} \mu(\widehat{\theta}^{\mathfrak{g}}(\gamma_1)\widehat{\theta}^{\mathfrak{g}}(\gamma_2)) &= \mu(\widehat{\theta}^{\mathfrak{g}}(\gamma_1\gamma_2)) = \gamma_1\gamma_2 \\ &= \mu\widehat{\theta}^{\mathfrak{g}}(\gamma_1)\mu\widehat{\theta}^{\mathfrak{g}}(\gamma_2) . \end{aligned}$$

Thus μ is an epimorphism of $\widehat{\theta}^{\mathfrak{g}}(K)^\wedge$ onto K^\wedge and $\mu\widehat{\theta}^{\mathfrak{g}}$ is the identity. It follows that μ is one-to-one on $\widehat{\theta}^{\mathfrak{g}}(K)$ and furthermore that $\widehat{\theta}^{\mathfrak{g}}\mu$ is the identity on $\widehat{\theta}^{\mathfrak{g}}K$: $\widehat{\theta}^{\mathfrak{g}}\mu(\widehat{\theta}^{\mathfrak{g}}(\gamma)) = \widehat{\theta}^{\mathfrak{g}}(\gamma)$.

Combining our results to this point we see that

$$\mathfrak{M}_D = \text{diag}(K^\wedge \times K^\wedge) \cong K^\wedge .$$

It follows that K is a reasonable candidate for the group \mathfrak{G} such that $D \cong L_1(\mathfrak{G})$. Indeed, if \mathfrak{G} is such a group then $\mathfrak{G}^\wedge = \mathfrak{M}_D$. Since $\mathfrak{M}_D = K^\wedge$, we conclude $\mathfrak{G} = K$.

We shall now define a map $T: D \rightarrow L_1(K)$. As usual T is defined on

$$\begin{aligned} \mathfrak{F} &\equiv F_{L_1(K)}(L_1(G), L_1(H)) \\ &= \left\{ f: f: L_1(G) \times L_1(H) \rightarrow L_1(K), \|f\| \right. \\ &\quad \left. \equiv \sum_{(a,b)} \|f(a, b)\| \|a\| \|b\| < \infty, f(0, b) = f(a, 0) + 0 \right\} \end{aligned}$$

[2, 3]. Thus if $c(a, b)$ is the function taking the value c at (a, b) we set

$$T(c(a, b)) = \int_{N^{\mathfrak{g}}} ca(\xi + \rho) d\rho * \int_{N^{\mathfrak{H}}} b(\eta + \sigma) d\sigma$$

where $N^{\mathfrak{H}} = \ker(\theta^{\mathfrak{H}})$. We note that each of the integrals above is a function on K and hence so is the indicated convolution. It is a simple matter to verify that when T is extended by linearity it is a

bounded epimorphism of the algebra \mathfrak{F} onto $L_1(K)$ and that T annihilates the reducing ideal I , modulo which the algebra \mathfrak{F} is D . (The surjectivity of T follows from the fact that the integrals $\int_{N^G} \equiv T_G$ and $\int_{N^H} \equiv T_H$ are epimorphisms, from a simple application of approximate identities and from P. J. Cohen's factorization theorem [1, 3, 4].)

We show now for T , which may be regarded as a mapping of D onto $L_1(K)$,

LEMMA 3.3. *T is an isomorphism if and only if D is semisimple.*

Proof. Clearly, if T is an isomorphism then D is semisimple.

Conversely, if D is semisimple and if $T(z) = 0$, where $z = \sum_{n=1}^\infty c_n(a_n \otimes b_n)$ [2, 3], then for any γ in K^\wedge , $T^\wedge(z)(\gamma) = 0$. Thus

$$\begin{aligned} T^\wedge(z)(\gamma) &= \sum_{n=1}^\infty T_G(\widehat{c_n a_n})(\gamma) T_H(\widehat{b_n})(\gamma) \\ &= \sum_{n=1}^\infty c_n^\wedge(\gamma) T_G(\widehat{a_n})(\gamma) T_H(\widehat{b_n})(\gamma) = 0. \end{aligned}$$

However,

$$\begin{aligned} T_G^\wedge(a)(\gamma) &= \int_K T_G(a)(\zeta) \overline{(\zeta, \gamma)} d\zeta \\ &= \int_K \left(\int_{N^G} a(\xi + \rho) d\rho \right) \overline{(\zeta, \gamma)} d\zeta \\ &= \int_K \left(\int_{N^G} a(\xi + \rho) \overline{(\xi + \rho, \gamma)} d\rho \right) d\zeta \\ &= a^\wedge(\alpha) \end{aligned}$$

where $\alpha = \widehat{\theta^G}(\gamma)$. After similar arguments about T_H we find

$$T^\wedge(z)(\gamma) = \sum_{n=1}^\infty c_n^\wedge(\gamma) a_n^\wedge(\alpha) b_n^\wedge(\beta)$$

where $\beta = \widehat{\theta^H}(\gamma)$. In other words $T^\wedge(z)(\gamma) = z^\wedge(\alpha, \beta)$ where $\mu(\alpha) = \gamma(\beta)$ and (α, β) corresponds to an element of \mathfrak{M}_D . Since $T^\wedge(z)(\gamma) \equiv 0$ for all γ , we find $z^\wedge(\alpha, \beta) \equiv 0$ for all (α, β) corresponding to elements of \mathfrak{M}_D . The semisimplicity assumption now shows $z = 0$ and hence that T is an isomorphism.

We now conclude by proving

LEMMA 3.4. *D is semisimple.*

Proof. Let z belong to the radical of D . As in [3, 4] we may assume that z is of the form $\sum_{n=1}^\infty c_n(a_n \otimes b_n)$ where, for fixed compact

sets U, V, W in $G^\wedge, H^\wedge, K^\wedge$ and for all n , support $a_n^\wedge(\alpha) \subset U$, support $b_n^\wedge(\beta) \subset V$, and support $c_n^\wedge(\gamma) \subset W$. Furthermore, we may assume that each c_n is of the form $c_{n_1} * c_{n_2} * c_{n_3}$ and thus in effect that

$$z = \sum_{n=1}^{\infty} c_{n_1}(c_{n_2} a_n \otimes c_{n_3} b_n)$$

where support $c_{n_1}^\wedge(\gamma) \subset W$.

Since $L_1(K)^G$ is an ideal in $L_1(G)$ and since there is a corresponding statement for $L_1(K)^H$, we conclude that there are elements d_{n_2}, d_{n_3} in $L_1(K)$ such that $\tilde{d}_{n_2}(\xi) = c_{n_2} a_n(\xi), \tilde{d}_{n_3}(\eta) = c_{n_3} b_n(\eta)$.

Furthermore, $\tilde{d}_{n_2}^\wedge(\alpha) = d_{n_2}(\mu(\alpha)), \tilde{d}_{n_3}^\wedge(\beta) = d_{n_3}(\nu(\beta))$, and $d_{n_2}^\wedge(\gamma) \neq 0$, or $d_{n_3}^\wedge(\gamma) \neq 0$ implies $d_{n_2}^\wedge(\mu\hat{\theta}^G(\gamma)) = \tilde{d}_{n_2}^\wedge(\hat{\theta}^G(\gamma)) \neq 0$, etc., i.e., that $\gamma \in \mu$ (support $\tilde{d}_{n_2}^\wedge$), etc. Thus there is a fixed compact set Y containing the supports of all $c_{n_1}^\wedge, c_{n_2}^\wedge, c_{n_3}^\wedge, d_{n_1}^\wedge, d_{n_2}^\wedge, d_{n_3}^\wedge$. Hence there is a fixed c in $L_1(K)$ such that $c^\wedge(\gamma) \equiv 1$ on Y , support $c^\wedge(\gamma)$ is compact and

$$0 \leq c^\wedge(\gamma) \leq 1.$$

For this c it is true that $c_{n_j} = c_{n_j} * c, d_{n_j} = d_{n_j} * c, j = 1, 2, 3$. Thus we find

$$\begin{aligned} z &= \sum_{n=1}^{\infty} c_n(a_n \otimes b_n) = \sum_{n=1}^{\infty} c_{n_1}(c_{n_2} a_n \otimes c_{n_3} b_n) \\ &= \sum_{n=1}^{\infty} c_{n_1}(d_{n_2} \otimes d_{n_3}) = \sum_{n=1}^{\infty} c_{n_1}(d_{n_2} c \otimes d_{n_3} c) \\ &= \left(\sum_{n=1}^{\infty} c_{n_1} d_{n_2} d_{n_3} \right) (c \otimes c). \end{aligned}$$

However, for all γ in K^\wedge

$$c_{n_1} \widehat{d_{n_2} d_{n_3}}(\gamma) = c_{n_1}^\wedge(\gamma) d_{n_2}^\wedge(\gamma) d_{n_3}^\wedge(\gamma).$$

Furthermore

$$\begin{aligned} d_{n_2}(\zeta) &= \int_G a_n(\xi) c_{n_2}(\zeta - \theta^G(\xi)) d\xi \\ d_{n_3}(\zeta) &= \int_H b_n(\eta) c_{n_3}(\zeta - \theta^H(\eta)) d\eta. \end{aligned}$$

Thus

$$\begin{aligned} d_{n_2}^\wedge(\gamma) &= \int_G \int_K a_n(\xi) c_{n_2}(\zeta - \theta^G(\xi)) \overline{(\zeta, \gamma)} d\xi d\zeta \\ &= \int_G \int_K a_n(\xi) c_{n_2}(\zeta_1) \overline{(\zeta_1, \gamma)} (\theta^G(\xi), \gamma) d\xi_1 d\zeta \\ &= a_n^\wedge(\hat{\theta}^G(\gamma)) c_{n_2}(\gamma) \end{aligned}$$

and similarly $d_{n_3}^\wedge(\gamma) = b_n^\wedge(\hat{\theta}^H(\gamma)) c_{n_3}(\gamma)$. We see then that

$$c_{n_1}^{\widehat{(\gamma)}} d_{n_2}^{\widehat{(\gamma)}} d_{n_3}^{\widehat{(\gamma)}} = c_{n_1}^{\widehat{(\gamma)}} c_{n_2}^{\widehat{(\gamma)}} a_n^{\widehat{(\theta^G(\gamma))}} c_{n_3}^{\widehat{(\gamma)}} b_n^{\widehat{(\theta^H(\gamma))}}$$

and since $\mu^{\widehat{\theta^G}(\gamma)} = \nu^{\widehat{\theta^H}(\gamma)} = \gamma$ we conclude that

$$\sum_{n=1}^{\infty} c_{n_1}^{\widehat{(\gamma)}} d_{n_2}^{\widehat{(\gamma)}} d_{n_3}^{\widehat{(\gamma)}} = z^{\widehat{(\{\theta^G(\gamma), \theta^H(\gamma)\})}}$$

which is zero as a consequence of our assumption. Thus $z = 0$ and the semisimplicity of D is established.

Hence, in the context indicated above and suggested by the diagram

$$\begin{array}{ccc} G & & H \\ & \searrow \theta^G & \swarrow \theta^H \\ & & K \end{array}$$

there obtains the formula

$$L_1(G) \otimes_{L_1(K)} L_1(H) \cong L_1(K) .$$

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