ON QUASI-ISOMORPHIC INVARIANTS OF PRIMARY GROUPS

PAUL DANIEL HILL
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Two primary groups $G$ and $H$ are quasi-isomorphic if there exist subgroups $G^*$ and $H^*$ of $G$ and $H$ such that $G^*$ and $H^*$ are isomorphic and such that $G/G^*$ and $H/H^*$ are bounded. The paper is concerned with properties, of primary groups, that are invariant under the relation of quasi-isomorphism. In the final section, a condition is given which is necessary and sufficient in order that the primary groups $G$ and $H$ be quasi-isomorphic in case $G$ and $H$ are both direct sums of closed groups.

The main result of the paper is that quasi-isomorphism commutes with direct decomposition for the class of primary groups whose first Ulm factors are direct sums of countable groups.

The connection between the relation of quasi-isomorphism of primary groups and their Ulm invariants was investigated by Beaumont and Pierce in [1] and [2] and by the author in [4]. Cutler [3] has recently studied properties of primary groups which are invariant under the relation of quasi-isomorphism. For example, it was proved in [3] that the property of being a direct sum of cyclic groups is invariant under quasi-isomorphism. Irwin and Richman proved, for primary groups, in [9] that the property of being a direct sum of countable groups is also a quasi-isomorphic invariant. We shall establish a decomposition theorem which contains these results as special cases; specifically, the following theorem is proved. Suppose that $G$ and $H$ are primary groups and suppose that $G = \sum G_i$ where $G_i/\mathfrak{p}G_i$ is a direct sum of cyclic groups. If $H$ is quasi-isomorphic to $G$, then $H = \sum H_i$ where $H_i$ is quasi-isomorphic to $G_i$. There is also a decomposition theorem proved for group pairs $(G, H)$ where $G$ is a direct sum of cyclic groups or a closed group and $H$ is a cobounded subgroup of $G$. In answer to a question in [3], we show that the property of being a direct sum of closed groups is not a quasi-isomorphic invariant. Even in one of the simplest cases, where $G = \bar{B} + B$, a group $H$ quasi-isomorphic to a direct sum $G$ of closed groups need not be a direct sum of closed groups. However, we show that if $G$ and $H$ are direct sums of closed groups, then $G$ and $H$ are quasi-isomorphic if they satisfy a condition which is obviously necessary; if for some bounded subgroup $B$ there exists an isomorphism from $(G + B)[\mathfrak{p}]$ onto $(H + B)[\mathfrak{p}]$ that does not alter heights more than a fixed positive
integer \( k \), then \( G \) and \( H \) are quasi-isomorphic.

Two abelian groups \( G \) and \( H \) are said to be quasi-isomorphic if there exist isomorphic subgroups \( H^* \) and \( G^* \) of \( H \) and \( G \), respectively, such that \( G/G^* \) and \( H/H^* \) are bounded. The notation \( G \cong H \) is used to mean that \( G \) and \( H \) are quasi-isomorphic. We shall call a subgroup \( A \) of \( G \) cobounded if \( G/A \) is bounded.

2. Cobounded subgroups inherit basic subgroups. Suppose that \( G \) is a primary group and that \( H \) is a subgroup of \( G \) such that \( H \cong p^nG \) for some positive integer \( n \). Let \( B \) be a basic subgroup of \( G \). The question was raised in [3] as to whether there exists a basic subgroup \( B' \) of \( H \) such that \( P^*B \subseteq B' \subseteq B \), and some partial results were obtained concerning the problem. Megibben has pointed out to the author that not only does such a \( B' \) always exists but that it is unique. In fact, the following theorem can readily be established. And we shall make use of it later on in the paper.

**Theorem 2.1.** Suppose that \( H \) is a cobounded subgroup of the primary group \( G \). If \( B \) is pure and dense in \( G \), relative to the \( p \)-adic topology, then \( B \cap H \) is pure and dense in \( H \).

**Proof.** There is, of course, no proper cobounded subgroup of a divisible group. Thus \( \{ B, H \}/B = G/B \) and \( \{ B, H \} = G \), so \( H/B \cap H \cong \{ B, H \}/B = G/B \) and \( B \cap H \) is dense in \( H \). Since \( H \) is cobounded and since \( B[p] \) is dense in \( G[p] \), it is immediate that \( (B \cap H)[p] \) is dense in \( H[p] \). In order to prove that \( B \cap H \) is pure in \( H \), it suffices, according to [6], to prove that \( B \cap H \) is a neat subgroup of \( H \). Suppose that \( ph = b \) where \( h \in H \) and \( b \in B \cap H \). Since \( B \) is pure, there is an element \( b' \in B \) such that \( pb' = b = ph \). Now \( (h - b') - b'' \in H \) for some \( b'' \in B[p] \) since \( B[p] \) is dense in \( G[p] \) and since \( H \) is cobounded. Observe that \( b' + b'' \in B \cap H \) and that \( p(b' + b'') = b \). This completes the proof of the theorem.

Recall that the final rank of a basic subgroup is called the critical number of a primary group. Since a cobounded subgroup has the same final rank as the group, an immediate consequence of Theorem 2.1 is the following corollary.

**Corollary 2.2.** The critical number of a primary group is a quasi-isomorphic invariant.

Theorem 2.1 also yields a refinement of Theorem 5.1 in [3].

**Corollary 2.3.** If \( H \) is a cobounded subgroup of the primary group \( G \) and \( N \) is high in \( G \), then \( M = H \cap N \) is high in \( H \).
Proof. Since \( N \) is high in \( G \), we have the relation \( G[p] = N[p] + G'[p] \) where \( G' = \bigcap_{n<\omega} p^n G \). Thus \( H[p] = M[p] + H'[p] \) since \( H' = G' \). Now the purity of \( M \) in \( H \) implies that \( M \) is a high subgroup of \( H \).

The question now arises as to whether for each pure and dense subgroup \( B \) of a cobounded subgroup \( H \) of the primary group \( G \) there exists a pure and dense subgroup \( A \) of \( G \) such that \( B = A \cap H \).

**Theorem 2.4.** If \( H \) is a cobounded subgroup of the primary group \( G \) and if \( B \) is pure and dense in \( H \), then there exists a pure and dense subgroup \( A \) of \( G \) such that \( B = A \cap H \).

**Proof.** Choose \( A \) maximal in \( G \) with respect to \( A \cap H = B \). Then \( A \) is neat in \( G \). We show that the socle of \( A \) is dense in \( G[p] \). Let \( x \in G[p] \). There is an element \( a \in A \) such that \( x = a + h \) where \( h \in H \). Since \( ph \in A \cap H = B \) and since \( B \) is pure, there exists \( b \in B \) such that \( ph = pb \). Thus

\[
x = (a + b) + (h - b) \in \{A[p], H[p]\} \subseteq \{A[p], B[p]\} \subseteq A[p].
\]

By Theorem 1 in [6], \( A \) is pure and dense in \( G \).

Observe that \( B = A \cap H \) is cobounded in \( A \) since \( H \) is cobounded in \( G \). It follows that \( B \) is a direct sum of cyclic groups if and only if \( A \) is a direct sum of cyclic groups. Hence we have the following corollary.

**Corollary 2.5.** Let \( H \) be a cobounded subgroup of the primary group \( G \). The correspondence \( B \rightarrow B \cap H \) is a function from the basic subgroups of \( G \) onto the basic subgroups of \( H \).

3. The decompositions theorems.

**Theorem 3.1.** Suppose that the primary group \( G \) is a direct sum of cyclic groups. If \( H \) is a cobounded subgroup of \( G \), then there exist a nonnegative integer \( k \) and a decomposition \( G = \sum_{i=0}^{k} G_i \) such that \( H \cong \sum_{i=0}^{k} p^i G_i \).

**Proof.** Let \( G[p] = P + H[p] \). Since \( P \) is a discrete subsocle [7] of \( G \), it supports a pure subgroup \( A \) of \( G \); indeed, \( P \) supports a \( p^i \)-bounded direct summand \( A \) of \( G \) if \( p^i(G/H) = 0 \). Let \( G = A + G' \) and let \( H' \) be the image of \( H \) under the natural projection of \( G \) onto \( G' \). Since \( H \cap A = 0 \), \( H' \) is isomorphic to \( H \). Furthermore, \( H'[p] = G'[p] \); hence it suffices to prove the theorem for the case \( H[p] = G'[p] \). Let \( S = H[p] = G[p] \). Since \( H \) is a \( p^i \)-cobounded subgroup of \( G \), the Ulm invariants of \( H \) and \( G \) are related by the inequalities:

\[
\sum_{n}^{m+k} f_n(j) \leq \sum_{n}^{m+r+k} f_{n+r}(j) \quad \text{and} \quad \sum_{n}^{m+r+k} f_{n+r+k}(j) \leq \sum_{n}^{m+k} f_{n}(j)
\]
for all \( n, r \geq 0 \), where \( f \) is the Ulm function. It follows from an obvious modification of Lemma 1 in [4] that there exist an automorphism \( \pi \) of \( S \) and decompositions \( S = \sum P_n = \sum Q_n \) such that the nonzero elements of \( P_n \) and \( Q_n \) have height \( n \) in \( H \) and \( G \), respectively, and such that for each \( x_n \in P_n \) the relation \( \pi(x_n) \in Q_{n+1} \) holds for some nonnegative \( i \leq k \). Hence there exist decompositions \( G = \sum_{i=0}^{k} G_i \) and \( H = \sum_{i=0}^{k} H_i \) of \( G \) and \( H \) such that \( H_i \cong p^i G_i \) for \( 0 \leq i \leq k \). Thus \( H \cong \sum_{i=0}^{k} p^i G_i \).

**Remark.** The isomorphism between \( H \) and \( \sum_{i=0}^{k} p^i G_i \) in Theorem 3.1 cannot be replaced by set theoretic equality. This can be demonstrated by very simple examples.

**Corollary 3.2.** If \( H \) is a cobounded subgroup of the closed group \( G \), then there exist a nonnegative integer \( k \) and a decomposition \( G = \sum_{i=0}^{k} G_i \) such that \( H \cong \sum_{i=0}^{k} p^i G_i \).

**Proof.** Let \( B \) be a basic subgroup of \( G \). It follows from Theorem 2.1 that \( B \cap H \) is a basic subgroup of the cobounded subgroup \( H \) of \( G \). Since \( B \cap H \) is a cobounded subgroup of \( B \), Theorem 3.1 implies that there exist a nonnegative integer \( k \) and a decomposition \( B = \sum_{i=0}^{k} B_i \) such that \( B \cap H \cong \sum_{i=0}^{k} p^i B_i \). Thus \( G = B = \sum_{i=0}^{k} B_i \) and \( H \cong \sum_{i=0}^{k} p^i B_i \) since \( G \) and \( H \) are closed.

Our main decomposition theorem concerning quasi-isomorphism is the following.

**Theorem 3.3.** Suppose for the primary group \( G \) that \( G = \sum_{i \in \Lambda} G_i \), where \( G_i/p^i G_i \) is a direct sum of countable groups for each \( \lambda \in \Lambda \). If \( H \) is quasi-isomorphic to \( G \), then \( H = \sum_{i \in \Lambda} H_i \) where \( H_i \) is quasi-isomorphic to \( G_i \) for each \( \lambda \in \Lambda \).

**Proof.** Since decompositions lift, for arbitrary \( G \), from \( p^i G \) to \( G \), it suffices to prove the theorem for the case that \( H \) is a cobounded subgroup of \( G \). Suppose that \( p^i G \subseteq H \subseteq G \). In case \( G \) is a direct sum of cyclic groups, the theorem follows from Theorem 3.1 and the isomorphic refinement theorem for direct sums of cyclic groups. In fact, since \( H \) is \( p^i \)-cobounded in \( G \), we can write \( H = \sum_{i \in \Lambda} H_i \) where \( H_i \) is isomorphic to a \( p^i \)-cobounded subgroup of \( G_i \). Thus, in the general case, we can write \( H/p^i H = \sum_{i \in \Lambda} H_i^\ast \) where \( H_i^\ast \) is isomorphic to a \( p^i \)-cobounded subgroup of \( G_i/p^i G_i \). For each \( \lambda \in \Lambda \), let \( G_i^\ast \) be a cobounded subgroup of \( G_i \) such that \( G_i^\ast \subseteq p^i G_i \subseteq H_i^\ast \). Set \( G^\ast = \sum_{i \in \Lambda} G_i^\ast \). Now \( p^i G^\ast = p^i G = p^i H \), and \( G^\ast/p^i G^\ast = G^\ast/p^i G = \sum_{i \in \Lambda} (G_i^\ast/p^i G_i) = \sum_{i \in \Lambda} H_i^\ast = H/p^i H \). It follows that \( H \cong G^\ast \) by the uniqueness theorem [8] of Hill and Megibben, and the theorem is proved.
DEFINITION 3.4. A primary group \( G \) is said to be a pillared group if \( G/p^\omega G \) is a direct sum of cyclic groups.

Since the property of being a direct sum of cyclic groups is a quasi-isomorphic invariant, the property of being a pillared group is a quasi-isomorphic invariant. Our Theorem 3.3 shows that for pillared groups quasi-isomorphism is compatible with direct decompositions. We conclude this section with the following consequence of Theorem 3.3.

COROLLARY 3.5. Suppose that \( G = \sum G_i \) is a pillared group. If \( H \) is quasi-isomorphic to \( G \), then \( H = \sum H_i \) where \( H_i \) is isomorphic to a direct summand of \( G_1 + C_2 \) and \( C_2 \) is a direct sum of cyclic groups.

4. Some quasi-isomorphic variants. As we have mentioned, it was established in [3] and [9] that the property of being a direct sum of cyclic groups and the property of being a direct sum of countable groups are invariant under the relation of quasi-isomorphism of primary groups; this is also an immediate consequence of Corollary 3.5. Cutler observed in [3] that the property of being a closed group is a quasi-isomorphic invariant, but he left open the following question. If \( G \) is a direct sum of closed groups and if \( H \) is quasi-isomorphic to \( G \), does \( H \) have to be a direct sum of closed groups? The next theorem shows that the answer is negative.

Let \( A = \sum \{a_n\} \) and \( B = \sum \{b_n\} \) be copies of the standard basic subgroup, that is, let \( \{a_n\} \) and \( \{b_n\} \) denote cyclic groups of order \( p^n \). Denote by \( \bar{A} \) the closed group \( \sum \{a_n\}_r \), the torsion completion of \( A \). We want to consider the group \( G = \bar{A} + B \) and a certain cobounded subgroup of \( G \).

THEOREM 4.1. The group \( G = \bar{A} + B \), where \( A \) and \( B \) are copies of the standard basic subgroup, has a cobounded subgroup \( H \) with the following properties.

1. \( H \) is not pure-complete.
2. \( H \) is not semi-complete.
3. \( H \) is not a direct sum of closed groups.

Proof. Define \( H = \{pG, a_1, a_{n+1} + b_n\}_{n<\omega} \). Let \( S = \bar{A}[p] \), and observe that \( S \subseteq H \). We show that \( S \) does not support a pure subgroup of \( H \). Assume that \( S \) does support a pure subgroup \( K \) of \( H \). Since each element of \( S \) has the same height in \( H \) as in \( G \), it follows that \( K \) is pure in \( G \). Since \( K[p] = \bar{A}[p], \) \( K \) is a closed group; hence \( K \) is a direct summand of \( G \). In fact, we have the decompositions \( G = K + B \) and \( H = K + (H \cap B) \). It is easily verified that \( H \cap B = pB, \)
so we have the equation $H = K + pB$. The defining equation for $H$ and the above decompositions imply that \( \{pK + pB, a, a_{n+1} + b_n\}_n = K + pB \). Thus, for each positive integer $n$,

$$p^n(a_{n+2} + b_{n+1}) = p^n k_{(n)} + p^{n+1} b_{(n)}$$

where $k_{(n)} \in K$ and $b_{(n)} \in B$.

Define $s(1) = 1$ and suppose that a positive integer $s(i)$ has been chosen for $i \leq n$ such that $s(1) < s(2) < \cdots < s(n)$. Choose $s(n+1) > s(n)$ such that $b(n) \in \sum_{i < s(n+1)} \{b_i\}$. Since $p\iota(p^n k_{(n)}) = 0$ for each $n$ and since $K$ is closed, $\sum p^{\iota(n)} k_{(a(n))}$ must converge in $K$. Since

$$p^n k_{(n)} = p^n a_{n+2} + (p^n b_{n+1} - p^{n+1} b_{(n)})$$

and since $G = \tilde{A} + B$, it follows that $\sum (p^{\iota(n)} b_{(s(n)+1)} - p^{\iota(n+1)} b_{(a)})$ must converge in $B$. However, this is impossible since, for each positive integer $n$, the projection of the limit onto $\{b_{(s(n)+1)}\}$ is $p^{\iota(n)} b_{(s(n)+1)} \neq 0$. Thus $S$ does not support a pure subgroup of $H$, and we have verified (1).

Assume that $H$ is a direct sum of closed groups. Then $H$ is a direct sum of a countable number of closed groups since $G$, and therefore $H$ by Theorem 2.1, has a countable basic subgroup. It follows from Theorem 5.6 in [7] that a direct sum of a countable number of closed groups is pure-complete. Since we have already verified (1), we conclude that $H$ is not a direct sum of closed groups. Furthermore, every semi-complete group [10] is a direct sum of closed groups, so the theorem is proved.

**Corollary 4.2.** The property of being pure-complete is not a quasi-isomorphic invariant.

**Corollary 4.3.** The property of being semi-complete is not a quasi-isomorphic invariant.

**Corollary 4.4.** The property of being a direct sum of closed groups is not a quasi-isomorphic invariant.

In view of Corollary 4.4, a natural question is: what are the groups that are quasi-isomorphic to direct sums of closed groups? In this connection, we make the following observation.

**Proposition 4.5.** If the primary group $G$ is quasi-isomorphic to a direct sum of closed groups, then $G[p] = \sum S_i$ where (1) $S_i$ is complete and (2) there exists a fixed positive integer $k$ such that height $(x_1 + x_2 + \cdots + x_n) \leq \text{height } (x_i) + k$ if $x_i \in S_i$ for distinct $\lambda_1, \lambda_2, \ldots, \lambda_n$. 

ON QUASI-ISOMORPHIC INVARIANTS OF PRIMARY GROUPS

Proof. Suppose that $G$ is quasi-isomorphic to a direct sum of closed groups. Then there are closed groups $H_i$ such that $G \cong \sum_{i \in I} H_i$ and such that there exists an isomorphism $\pi$ from $G[p]$ onto $\sum_{i \in I} H_i[p]$ that does not alter heights (computed in $G$ and $H = \sum H_i$) more than a fixed positive integer $k$. Defining $S_i$ by the equation $\pi(S_i) = H_i[p]$, we have $S_i$'s which satisfy the conditions (1) and (2).

5. Quasi-isomorphism of direct sums of closed groups. Although two primary groups $G$ and $H$ can be quasi-isomorphic with one a direct sum of closed groups and the other not, there is a particularly simple criterion which determines whether $G$ and $H$ are quasi-isomorphic in case both $G$ and $H$ are direct sums of closed groups.

THEOREM 5.1. Suppose that the primary groups $G$ and $H$ are direct sums of closed groups. Then $G$ and $H$ are quasi-isomorphic if and only if there is a bounded group $B$ and an isomorphism between $(G + B)[p]$ and $(H + B)[p]$ that does not alter heights more than a fixed positive integer $k$.

Proof. If $G \cong H$, there exist $p^k$-cobounded subgroups $G^*$ and $H^*$ of $G$ and $H$, respectively, such that $G^* \cong H^*$. Define $\pi$ from $G^*[p]$ onto $H^*[p]$ as the restriction of some isomorphism $\varphi$ from $G^*$ onto $H^*$. Let $G[p] = P + G^*[p]$ and $H[p] = Q + H^*[p]$. The height of a nonzero element in $P$ or $Q$ does not exceed $k$. For a sufficiently large $p$-bounded group $B, |P + B| = |Q + B|$ and $\pi$ can be extended to an isomorphism from $(G + B)[p]$ onto $(H + B)[p]$ having the desired property that heights are not altered more than $k$.

For the proof of the nontrivial half of the theorem, we may assume that $B = 0$ since the relation of quasi-isomorphism is transitive. Thus suppose that $G = \sum G_\alpha$ and $H = \sum H_\mu$ are direct sums of closed groups and that $\pi$ is an isomorphism from $G[p]$ onto $H[p]$ that does not alter heights more than $k$. We wish to show that $G \cong H$. Since $G_\alpha$ or $H_\mu$ can be zero, there is no loss of generality in assuming that $\Lambda = M$. Thus we shall make this assumption. If $\Lambda$ is countable, then $G + H$ is pure-complete and the proof that $G \cong H$ is essentially the same as the proof of Corollary 1 in [4]. Let $K$ be a pure subgroup of $G + H$ such that $K[p] = \{(x, \pi(x)) : x \in G[p]\}$. Then $K$ is a subdirect sum of isomorphic cobounded subgroups $G^*$ and $H^*$ of $G$ and $H$, respectively.

We now assume that $\Lambda$ is uncountable and proceed by induction on the cardinality of $\Lambda$. According to the next theorem, there are decompositions $G = \sum G^*_\alpha$ and $H = \sum H^*_\lambda$ of $G$ and $H$ into closed groups such that, for each $\lambda \in \Lambda$, there exists a countable subset $M_\lambda$...
of $A$ having the property that
\[ \varphi(G_\lambda[p]) \subseteq \sum_{i \in I} H_\mu \quad \text{and} \quad \varphi^{-1}(H_\lambda[p]) \subseteq \sum_{i \in I} G_\mu \]
for some isomorphism $\varphi$ between $G[p]$ and $H[p]$ that alters heights no more than $k$. It follows that there is an ascending chain
\[ A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots , \quad A_\alpha \subseteq A , \]
that leads up to $A$ such that $\varphi(\sum_{\lambda \in \Lambda} G_\lambda[p]) = \sum_{\lambda \in \Lambda} H_\lambda[p]$ and $|A_\alpha| < |A|$. We conclude that there exist decompositions
\[ G = \sum_{\alpha} \sum_{\lambda \in I} G_\lambda^\alpha \quad \text{and} \quad H = \sum_{\alpha} \sum_{\lambda \in I} H_\lambda^\alpha \]
such that there is an isomorphism from $\sum_{\lambda \in I} G_\lambda^\alpha[p]$ onto $\sum_{\lambda \in I} H_\lambda^\alpha$ that does not alter heights more than $k$ and such that $|I_\alpha| < |A|$. The proof of the theorem is finished by an application of the induction hypothesis; however, we owe a proof of the following theorem.

**Theorem 5.2.** For the primary groups $G$ and $H$, suppose that $G = \sum_{i \in I} G_i$ and $H = \sum_{i \in I} H_i$ where $G_i$ and $H_i$ are closed groups. If $\pi$ is an isomorphism from $G[p]$ onto $H[p]$ that alters heights no more than $k$, then there are decompositions $G = \sum_{i \in I} G_i^\alpha$ and $H = \sum_{i \in I} H_i^\alpha$ of $G$ and $H$ into closed groups and an isomorphism $\varphi$ from $G[p]$ onto $H[p]$ that alters heights no more than $k$ such that, for each $\lambda \in \Lambda$, there exists a countable subset $M_\lambda$ of $\Lambda$ such that
\[ \varphi(G_\lambda^\alpha[p]) \subseteq \sum_{M_\lambda} H_\mu \quad \text{and} \quad \varphi^{-1}(H_\mu[p]) \subseteq \sum_{M_\lambda} G_\mu . \]

**Proof.** The proof is similar to the proof of Theorem 2 in [5], except in the present case most of the details are simpler. Here, we include only an outline of the proof. The following lemma is essential.

**Lemma 5.3.** Suppose that $G = B + \sum_{i \in I} G_i$ where $B$ is a direct sum of cyclic groups and $G_i$ is closed for each $\lambda \in \Lambda$. If $H$ is a closed group and if $\pi$ is an isomorphism from $H[p]$ into $G[p]$ that does not decrease heights more than a fixed positive integer $k$, then there exists a positive integer $n$ such that $\pi(p^n H \cap H[p])$ is contained in a finite number of the groups $G_i$.

Applying Lemma 5.3 to $\pi$ and $\pi^{-1}$ and working back and forth between $G_i$ and $H_i$, we obtain decompositions $G = A + \sum_{i \in I} G_i$ and $H = B + \sum_{i \in I} H_i$ such that (i) $A$ and $B$ are direct sums of cyclic groups, (ii) $\pi(\sum_{i \in I} G_i[p]) = \sum_{i \in I} H_i[p]$, (iii) For each $\lambda$, there exists a finite subset $M_\lambda$ of $\Lambda$ such that
\[ \pi(G_\lambda[p]) \subseteq \sum_{M_\lambda} H_\mu \quad \text{and} \quad \pi^{-1}(H_\mu[p]) \subseteq \sum_{M_\lambda} G_\mu . \]
(iv) there is an isomorphism $\varphi$ from $A[p]$ onto $B[p]$ that alters heights no more than $k$. This essentially finishes the proof of Theorem 5.2.

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Lowell Wayne Beineke, Frank Harary and Michael David Plummer, *On the critical lines of a graph* ................................................. 205
Larry Eugene Bobisud, *On the behavior of the solution of the telegraphist’s equation for large velocities* ............................................. 213
Richard Thomas Bumby, *Irreducible integers in Galois extensions* ....... 221
Chong-Yun Chao, *A nonimbedding theorem of nilpotent Lie algebras* ...... 231
Peter Crawley, *Abelian p-groups determined by their Ulm sequences* ...... 235
Bernard Russel Gelbaum, *Tensor products of group algebras* ............... 241
Newton Seymour Hawley, *Weierstrass points of plane domains* .......... 251
Paul Daniel Hill, *On quasi-isomorphic invariants of primary groups* ..... 257
Melvyn Klein, *Estimates for the transfinite diameter with applications to conformal mapping* ................................................. 267
Frederick M. Lister, *Simplifying intersections of disks in Bing’s side approximation theorem* ................................................. 281
Charles Wisson McArthur, *On a theorem of Orlicz and Pettis* ............. 297
Harry Wright McLaughlin and Frederic Thomas Metcalf, *An inequality for generalized means* ................................................. 303
Daniel Russell McMillan, Jr., *Some topological properties of piercing points* ......................................................... 313
Peter Don Morris and Daniel Eliot Wulbert, *Functional representation of topological algebras* ................................................. 323
Roger Wolcott Richardson, Jr., *On the rigidity of semi-direct products of Lie algebras* ......................................................... 339
Jack Segal and Edward Sandusky Thomas, Jr., *Isomorphic cone-complexes* ......................................................... 345
Richard R. Tucker, *The δ2-process and related topics* .......................... 349
David Vere-Jones, *Ergodic properties of nonnegative matrices. I* ........ 361