ESTIMATES FOR THE TRANSFINITE DIAMETER WITH APPLICATIONS TO CONFORMAL MAPPING

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Let \( f(z) \) be a member of the family \( S \) of functions regular and univalent in the open unit disk whose Taylor expansion is of the form: \( f(z) = z + a_2z^2 + \cdots \). Let \( D_w \) be the image of the unit disk under the mapping: \( w = f(z) \). An inequality for the transfinite diameter of \( n \) compact sets in the plane \( \{ T_i \} \) is established, generalizing a result of Renngli:

\[
d(T_1 \cap T_2) \cdot d(T_1 \cup T_2) \leq d(T_1) \cdot d(T_2).
\]

This inequality is applied to derive covering theorems for \( D_w \) relative to a class of curves issuing from \( w = 0 \), arcs on the circle: \( |w| = R \) as well as other point sets.

I. Preliminary considerations.

DEFINITION (1.1). Let \( E \) be a compact set in the plane. Set:

\[
V(z_1, \ldots, z_n) = \prod_{k=1}^{n} (z_k - z_i) \quad n \geq 2, \quad z_i \in E,
\]

\[
V_n = V_n(E) = \max_{z_1, \ldots, z_n \in E} |V(z_1, \ldots, z_n)|
\]

and

\[
d_n = d_n(E) = V_n^{2/(n-1)}.
\]

The transfinite diameter of \( E \) is then defined by:

\[
d = d(E) = \lim_{n \to \infty} d_n.
\]

A full discussion of the transfinite diameter and related constants can be found in [2, Chapter 7].

The following is a theorem of Hayman [3]:

THEOREM (1.2). Suppose \( f(z) \) is a function meromorphic in the unit disk with a simple pole of residue \( k \) at the origin, i.e., the expansion of \( f(z) \) about the origin is of the form:

\[
f(z) = \frac{k}{z} + a_0 + a_2z + \cdots.
\]

Let \( D_w \) denote the image of \( |z| < 1 \) under the mapping \( w = f(z) \) and let \( E_w \) denote the complement of \( D_w \) in the \( w \)-plane. Then:

\[
d(E_w) \leq k
\]

with equality if and only if \( f(z) \) is univalent.

Using Hayman's theorem is easy to prove the following:
THEOREM (1.3). Let \( w(z) = kz + a_z z^2 + a_{z^2} z^3 + \cdots \) be a function univalent in \( |z| < 1 \) and \( D_w \) the image of \( |z| < 1 \) under \( w(z) \). Then the complement of the image of \( D_w \) under the mapping: \( \zeta = 1/w \), which we denote by \( E_\zeta \), has transfinite diameter: \( 1/k \). In particular, if \( w(z) = z + a_z z^2 + \cdots \) then \( d(E_\zeta) = 1 \).

We will need to know the transfinite diameter of several specific sets.

LEMMA (1.4). Let \( E \) be the set union of:

(i) an arc of central angle \( \theta \), \( 0 \leq \theta \leq 2\pi \) lying on \( |w| = 1 \) with midpoint: \( w = 1 \).

(ii) a linear segment \([a, b] \), \( 0 \leq a \leq 1 \leq b \). Then the transfinite diameter of \( E \) expressed as a function of \( a, b \) and \( \theta \) is given by

\[
d(E) = \frac{\cos^2 \frac{\theta}{4} \left[ (1 + b) \left( 1 + a^2 - 2a \cos \frac{\theta}{2} \right)^{\frac{1}{2}} + (1 + a) \left( 1 + b^2 - 2b \cos \frac{\theta}{2} \right)^{\frac{1}{2}} \right]}{2 \left[ (1 + a) + (1 + a^2 - 2a \cos \frac{\theta}{2})^{\frac{1}{2}} \right] \times \left[ (1 + b) - (1 + b^2 - 2b \cos \frac{\theta}{2})^{\frac{1}{2}} \right]}
\]

where positive roots are taken throughout.

Proof. A univalent mapping, \( w = f(z) \), of \( |z| < 1 \) onto the complement of \( E \) with a simple pole at \( z = 0 \) will be constructed. According to Theorem (1.2) the residue of the mapping function is the transfinite diameter of \( E \). Define:

\[
w_1(z) = (z + \alpha)/(1 + \alpha z)
\]

where:

\[
\alpha = \frac{d - c + \csc \frac{\theta}{4}}{c} - \left[ \left( \frac{d - c + \csc \frac{\theta}{4}}{4} \right)^2 - 1 \right]^{\frac{1}{2}},
\]

\( d > 1 , \ 2c - d > 0 \).

Define:

\[
w_2 = \frac{1}{2} \left( w_1 + \frac{1}{w_1} \right) \quad w_3 = c(w_2 + 1) - d \]

\[
w_4 = (w_3^2 - 1)^{\frac{1}{2}} \quad w_5 = \frac{\cot \frac{\theta}{4} + w_4}{\cot \frac{\theta}{4} - w_4}.
\]
The composition of these five mappings is given by:

\[
\begin{aligned}
w(z) &= \cot \theta / 4 + \left\{ \frac{1}{2} c \left( \frac{z + \alpha}{1 + \alpha z} + \frac{1 + \alpha z}{z + \alpha} + 2 \right) - d \right\}^2 - 1 \right\}^{1/2} \\
&= \cot \frac{\theta}{4} - \left\{ \frac{1}{2} c \left( \frac{z + \alpha}{1 + \alpha z} + \frac{1 + \alpha z}{z + \alpha} + 2 \right) - d \right\}^2 - 1 \right\}^{1/2}.
\end{aligned}
\]

\(w(z)\) maps \( |z| < 1 \) onto the exterior of \( E \) (upon proper choice of the parameters \( c \) and \( d \), to be made presently); it has a simple pole at the origin of residue:

\[
c = \frac{\csc \theta / 4 + 2(d - c) \sec \theta / 4 + \tan \theta / 4 \sec \theta / 4 (d^2 + 1 - 2cd)}{(a + 1) \sin \theta / 4}.
\]

This is the transfinite diameter of \( E \). To express it in terms of \( a, b \) and \( \theta \) we note that the point \( w = b \) is the image of \( w_z = 1 \), and the point \( w = a \) is the image of \( w_z = -1 \). Using this to solve for \( c \) and \( d \) we find:

\[
d = \left[ a^2 + 1 - 2a \cos \frac{\theta}{2} \right]^{1/2}
\]

\[
c = \frac{\left[ a^2 + 1 - 2a \cos \frac{\theta}{2} \right]^{1/2}}{(a + 1) \sin \theta / 4} + \frac{\left[ b^2 + 1 - 2b \cos \frac{\theta}{2} \right]^{1/2}}{2(b + 1) \sin \theta / 4}.
\]

Substituting these values in the above expression for the residue we arrive at the expression given in the statement of the lemma.

When \( a = b = 1 \) the set \( E \) is simply an arc of central angle \( \theta \) on the unit circle. Using the lemma we find: \( d(1, 1, \theta) = \sin \theta / 4 \).

**Lemma (1.5).** Let \( E \) be the set union of two linear segments issuing from the origin at an angle \( 2\pi \alpha \), \( 0 < \alpha \leq 1/2 \), each of length: \( 4\alpha (1 - \alpha)^{1-\alpha} \). Then: \( d(E) = 1 \).

**Proof.** The mapping of \( |z| < 1 \) onto the exterior of \( E \) is given by the Schwarz-Christoffel formula:

\[
w = c \cdot \int_{0}^{z} \frac{(z + 1) \left( z - 1 \right)^{2a-1} \left( z - 1 + 2\alpha - 2[\alpha^2 - \alpha]^{1/2} \right)}{z^2} \times \frac{(z - 1 + 2\alpha + 2[\alpha^2 - \alpha]^{1/2})}{dz}
\]

\[
= c \cdot \frac{(z + 1)^{2-2a} (z - 1)^{2a}}{z}.
\]
The residue of this function (the transfinite diameter of $E$) is $c$. Noting that the map carries $z = 1 - 2\alpha + 2(\alpha^2 - \alpha)^{1/2}$ onto $w = 4\alpha^a(1 - \alpha)^{1-a}e^{ia}$ we find that $d(E) = |c| = |e^{i\pi a}/(-1)^a| = 1$.

Finally, we describe two types of symmetrization.

Steiner symmetrization of a plane set $E$ with respect to a straight line $l$ in the plane transforms $E$ into a set $E'$ characterized by the following:

(i) $E'$ is symmetric with respect to $l$.

(ii) Any straight line orthogonal to $l$ that intersects one of the sets $E$ or $E'$ also intersects the other. Both intersections have the same linear measure, and

(iii) The intersection with $E'$ consists of just one line segment, and may degenerate to a point.

Circular symmetrization of a plane set $E$ with respect to the positive real axis transforms $E$ into a set $E'$ characterized by the following:

(i) $E'$ is symmetric with respect to the real axis.

(ii) Any circle $|z| = r$, $0 \leq r < \infty$ that intersects one of the sets $E$ or $E'$ also intersects the other. Both intersections have the same linear measure, and

(iii) The intersection with $E'$ consists of just one arc with its midpoint on the positive real axis, and may degenerate to a point.

The following theorem describes the effect of these symmetrizations on the transfinite diameter [5; p. 6 and Note A]:

**Theorem (1.6).** Neither Steiner nor circular symmetrization increase the transfinite diameter.

### II. Estimates for the transfinite diameter.

A recent result of Renngli [6] is the following:

**Theorem (2.1).** If $T_1$ and $T_2$ are compact sets in the plane, then

\[ d(T_1 \cup T_2) \cdot d(T_1 \cap T_2) \leq d(T_1) \cdot d(T_2). \]

We will now generalize this to obtain an inequality for $n$ compact sets.

**Theorem (2.2).** If $T_1, T_2, \ldots, T_n$ are compact sets in the plane, let $C_k$ be the set of all points contained in at least $k$ of the $T_j$’s. Then:

\[ \prod_{k=1}^{n} d(C_k) \leq \prod_{k=1}^{n} d(T_k). \]
Proof. For \( n = 1 \) this is a triviality. For \( n = 2 \) it is identical with Renngli’s result:

\[
d(T_1 \cup T_2) \cdot d(T_1 \cap T_2) \leq d(T_1) \cdot d(T_2).
\]

Suppose the theorem is already established for \( n - 1 \) sets. Let \( B_n \) be the set of all points lying in at least \( k \) of the sets \( T_1, T_2, \ldots, T_{n-1} \). Obviously: \( B_{n-1} \subseteq B_{n-2} \subseteq \cdots \subseteq B_1 \). Also:

\[
C_n = B_{n-1} \cap T_n, \quad C_1 = B_1 \cup T_n,
\]

\[
C_k = B_k \cup \{B_{k-1} \cap T_n\} \quad (k = 2, 3, \ldots, n - 1).
\]

If \( d(B_{n-1} \cap T_n) = d(C_n) = 0 \), (1) is certainly true.

If \( d(B_{n-1} \cap T_n) \neq 0 \), then, a fortiori,

\[
d(B_k \cap T_n) \neq 0 \quad (k = 1, 2, \ldots, n - 1).
\]

By (2), (3) and Renngli’s inequality:

\[
d(C_n) = d(B_{n-1} \cap T_n)
\]

\[
d(C_k) \cdot d(B_k \cap T_n) = d(C_k) \cdot d(B_k \cap B_{k-1} \cap T_n) \leq d(B_k) \cdot d(B_{k-1} \cap T_n)
\]

\[
(k = 2, \ldots, n - 1)
\]

\[
d(C_1) \cdot d(B_1 \cap T_n) \leq d(B_1) \cdot d(T_n).
\]

Multiplying these inequalities and dividing both sides by \( \prod_{k=1}^{n} d(B_k \cap T_n) \) yields

\[
\prod_{k=1}^{n} d(C_k) \leq \prod_{k=1}^{n-1} d(B_k) d(T_n)
\]

and the theorem is proved, since by the induction hypothesis

\[
\prod_{k=1}^{n-1} d(B_k) \leq \prod_{k=1}^{n-1} d(T_k).
\]

DEFINITION (2.3). A point set \( T \) will be called a broken ray provided

(i) for every \( r \geq 0 \) there is a point \( z \in T \) such that: \( |z| = r \).

(ii) the set of numbers \( r \geq 0 \) for which there is more than one point \( z \in T \) such that: \( |z| = r \) is a set of measure zero.

DEFINITION (2.4). Let \( T \) be a subset of a broken ray. The point sets: \( \eta_1 T, \eta_2 T, \ldots, \eta_n T \) where \( \{\eta_k\}^n \) are the \( n \)-th roots of unity, will be called symmetric images of \( T \). The point set: \( \bigcup_{k=1}^{n} \eta_k \cdot T \) will be called the set of \( n \)-fold symmetry generated by \( T \) and will be denoted by \( T^{(n)} \). Subsets of \( T^{(n)} \) will be denoted by \( \tilde{T}^{(n)} \).
DEFINITION (2.5). Let \( T \) be a subset of a broken ray, \( T^{(n)} \) the set of \( n \)-fold symmetry generated by \( T \) and \( \bar{T}^{(n)} \) a subset of \( T^{(n)} \). We define the circular projection of \( \bar{T}^{(n)} \) as a subset, \( \bar{\tau}^{(n)} \), of the set of \( n \)-fold symmetry, \( \tau^{(n)} \), generated by the positive real axis, \( \tau \). A point \( z = \eta_k \cdot r \) will belong to the projection \( \bar{\tau}^{(n)} \) if and only if there is a point: \( \zeta \in \eta_k \cdot T \cap \bar{T}^{(n)} \) such that \( |\zeta| = r \).

DEFINITION (2.6). Let \( \bar{\tau}^{(n)} \) be a set such as described in definition (2.5). We will use the symbol \( l_k \) to denote the measure of the set of real numbers \( r \), \( 0 \leq r < \infty \) such that at least \( k \) of the symmetric images of \( r \) lie in \( \bar{\tau}^{(n)} \).

REMARK (2.7). Let \( L \) denote the linear measure of \( \bar{\tau}^{(n)} \); that is, the sum of the linear measures of the \( n \) legs of \( \bar{\tau}^{(n)} \). Then
\[
\sum_{k=1}^{n} l_k = L.
\]

The reason is that if \( I \) is a set of real numbers which have symmetric images on exactly \( k \) legs of \( \bar{\tau}^{(n)} \) the measure of \( I \) is included in: \( l_1, l_2, \ldots, l_k \), that is, it is counted \( k \) times in: \( \sum_{k=1}^{n} l_k \).

The following theorem of Fekete is essential to our work [2; page 259].

THEOREM (2.8). Let \( E \) be a compact set and \( p(z) \) a polynomial of degree \( n \):
\[
p(z) = z^n + c_1 z^{n-1} + \cdots + c_n.
\]

Let \( E_0 \) be the set of all points \( z \) such that \( p(z) \) lies in \( E \); we will call \( E_0 \) a root set of \( E \). Then: \( d(E_0) = d(E)^{1/n} \).

THEOREM (2.9). Suppose \( \bar{T}^{(n)} \) is a subset of a set of \( n \)-fold symmetry with: \( d(\bar{T}^{(n)}) = 1 \), and \( \bar{\tau}^{(n)} \) its circular projection. If \( l_k \) (\( k = 1, 2, \ldots, n \)) represent the measures defined in (2.6), then:
\[
\prod_{k=1}^{n} l_k \leq 4.
\]

Equality occurs when \( \bar{T}^{(n)} \) is itself a set of \( n \)-fold symmetry, consisting of a single component and identical with its circular projection: \( \bar{T}^{(n)} = \bar{\tau}^{(n)} \).

Proof. Let \( T_k = \eta_k \cdot \bar{T}^{(n)} \), (\( k = 1, 2, \ldots, n \)). Clearly:
\[
(4) \quad d(T_k) = d(\bar{T}^{(n)}) = 1 \quad (k = 1, 2, \ldots, n)
\]
since the transfinite diameter is unaffected by rigid motions.
Let $C_k$ be the set of all points contained in at least $k$ of the $T_i$'s; that is, the set of all points $z$ such that at least $k$ of the symmetric images of $z$ lie in $\tilde{T}^{(n)}$. Each of the sets $C_k$ is a set of $n$-fold symmetry.

Let $\gamma_k$ be the circular projection of $C_k$. In view of our description of the sets $C_k$ it is not difficult to see that the measure of a leg of $\gamma_k$ is $l_k$.

Let $B_k$ be the set of which $C_k$ is the root set with respect to the polynomial $p(z) = z^n$. Since $C_k$ is a set of $n$-fold symmetry $B_k$ is a subset of a single broken ray. Let $\beta_k$ be the set of which $\gamma_k$ is the root set with respect to the polynomial $p(z) = z^n$. As above, $\beta_k$ will be a subset of a single broken ray; in this case the positive real axis.

Since $\gamma_k$ is the circular projection of $C_k$ it follows that $\beta_k$ is the circular projection of $B_k$. When $n = 1$ circular projection is the same transformation as circular symmetrization. Therefore:

$$d(C_k) = d(B_k)^{1/n}$$

by Theorem (2.8)

$$\geq d(\beta_k)^{1/n}$$

by Theorem (1.6)

$$\geq \left[ \frac{(l_k)^n}{4} \right]^{1/n} = \frac{l_k}{\sqrt[n]{4}}$$

since $\beta_k$ has linear measure no less than: $(l_k)^n$. So finally we have:

$$1 = d(\tilde{T}^{(n)}) = \prod_{k=1}^{n} d(T_k)$$

by (4)

$$\geq \prod_{k=1}^{n} d(C_k)$$

by Theorem (2.2)

$$\geq \prod_{k=1}^{n} \frac{l_k}{\sqrt[n]{4}} = \frac{1}{4} \prod_{k=1}^{n} l_k$$

by (5).

This is the desired result: $4 \geq \prod_{k=1}^{n} l_k$.

This theorem contains as a special case a result of G. Szegö [7]; in our notation his result reads: Suppose that $\tilde{T}^{(n)} = \tilde{\tau}^{(n)}$ (i.e., it consists of straight line segments) and that $\tilde{T}^{(n)}$ is a connected set. Then $\prod_{k=1}^{n} L_k \leq 4$ where $L_k$ is the linear measure of the $k$-th leg of $\tilde{T}^{(n)}$, ($k = 1, 2, \cdots, n$).

Proof. In this case: $L_k = l_k$.

The next theorem establishes bounds on the content of a set lying on a circle as a function of the radius and the transfinite diameter of the set.

THEOREM (2.10). Let $A'_1, A'_2, \cdots, A'_s, A'_s \subseteq A'_{s+1}$ be a nested sequence of arcs on the circle $|z| = R$ where the central angle swept out by
A_k' is \( \theta_k, \, 0 < \theta_k \leq 2\pi/n \). Let \( \eta_1, \eta_2, \cdots, \eta_n \) denote the \( n \)-th roots of unity and let \( \alpha(i) \) be a mapping of the set of integers \( \{1, 2, \cdots, n\} \) onto itself. Define:

\[
A_k = \eta_{\alpha(k)}A_k' \quad (k = 1, 2, \cdots, n)
\]

and let: \( A = A_1 \cup A_2 \cup \cdots \cup A_n \). Then:

\[
\prod_{k=1}^n \sin \frac{n\theta_k}{4} \leq \left[ \frac{d(A)}{R} \right]^{n^2}.
\]

**Proof.** \( d(A) = d(\eta_k \cdot A) \) \((k = 1, 2, \cdots, n)\). Therefore:

(6) \[
[d(A)]^n = \prod_{k=1}^n d(\eta_k \cdot A).
\]

Let \( C_k \) be the set of all points contained in at least \( k \) of the sets: \( \eta_j \cdot A \). It follows from our hypothesis that the sets \( A_k' \) are nested that:

\[
C_k = \eta_1 \cdot A_k \cup \eta_2 \cdot A_k \cup \cdots \cup \eta_n \cdot A_k
\]

for each \( k, \, 1 \leq k \leq n \). Thus \( C_k \) is the root set with respect to the polynomial \( w(z) = z^n \) of an arc on the circle \( |w| = R^* \) of central angle \( n \cdot \theta_k \). The transfinite diameter of such an arc is, by virtue of the equality: \( d(c \cdot E) = |c| \cdot d(E) \) \((c \text{ a constant})\) given by: \( R^* \cdot \sin (n \cdot \theta_k/4) \). Therefore by Theorem (2.8):

(7) \[
\prod_{k=1}^n d(C_k) = (R^* \cdot \sin (n \theta_k/4))^{1/n}.
\]

Also, by virtue of Theorem (2.2) we have that:

(8) \[
\prod_{k=1}^n d(C_k) \geq \prod_{k=1}^n d(C_k).
\]

Combining inequalities (6), (7) and (8) we conclude:

\[
[d(A)]^n \geq \prod_{k=1}^n [R^* \cdot \sin (n \theta_k/4)]^{1/n}
\]

or

\[
[d(A)/R]^n \geq \prod_{k=1}^n \sin (n\theta_k/4)
\]

as claimed.

**III. Covering theorems.** The class of functions regular and univalent in \( |z| < 1 \) whose expansion is of the form: \( f(z) = z + a_2z^2 + \cdots \) will be denoted by \( S \). Let \( D_w \) be the image of the unit disk under the mapping \( w = f(z) \in S \). A classical result of Koebe and Bieberbach states that \( D_w \) contains the disk \( |w| < 1/4 \) irrespective of the mapping
function \( w = f(z) \) [2; page 41]. G. Szego later noted that [8]: If \( \alpha, \beta \) are two values lying in the complement of \( D_w \) and if the segment connecting \( \alpha \) and \( \beta \) passes through the origin, then: \(|\alpha| + |\beta| \geq 1\).

Generalizing these results, Michael Fekete made the following conjecture: Given \( n \) rays issuing from the origin \( w = 0 \) at equal angles \( 2\pi/n \), let \( L \) denote the linear measure of the intersection of these rays with \( D_w \). Then: \( L \geq n \cdot \sqrt[4]{1/4} \). The theorems of Koebe-Bieberbach and Szego are the cases \( n = 1 \) and \( n = 2 \). For arbitrary \( n \) the inequality was proved in 1964 by Marcus [4].

Our first theorem in this section further generalizes these results by considering a more general class of curves issuing from the origin in place of the \( n \) rays of Fekete's conjecture. The results of the preceding section will be used to prove this as well as various other covering theorems for the class \( S \).

**Theorem (3.1).** Let \( f(z) \in S \) and let \( D_w \) be the image of the disk \( |z| < 1 \) under the mapping \( w = f(z) \). Let \( S^{(n)} \) be a set of \( n \)-fold symmetry generated by an arbitrary broken ray; \( \tilde{S}^{(n)} \), a subset of \( S^{(n)} \) defined by: \( \tilde{S}^{(n)} = D_w \cap S^{(n)} \) and \( \tilde{\sigma}^{(n)} \) the circular projection of \( \tilde{S}^{(n)} \). Denote by \( L \) the linear measure of \( \tilde{\sigma}^{(n)} \). Then \( L \geq n \cdot \sqrt[4]{1/4} \).

**Proof.** Let \( E_{\zeta} \) represent the image of the complement of \( D_w \) under the transformation: \( \zeta = 1/w \). Then by Theorem (1.3) it follows that: \( d(E_{\zeta}) = 1 \). Let \( T^{(n)} \) denote the set of \( n \)-fold symmetry that is the image of \( S^{(n)} \) under the transformation \( \zeta = 1/w \) and let \( \tilde{T}^{(n)} \) denote the subset of \( T^{(n)} \) defined by: \( \tilde{T}^{(n)} = E_{\zeta} \cap T^{(n)} \). Denote by \( \tilde{\zeta}^{(n)} \) the circular projection of \( \tilde{T}^{(n)} \). It is clear from the definition of the sets involved that \( \tilde{T}^{(n)} \) is the complement with respect to \( T^{(n)} \) of the image of \( \tilde{S}^{(n)} \) under the transformation \( \zeta = 1/w \) and consequently, that \( \tilde{\zeta}^{(n)} \) is the complement with respect to \( \tau^{(n)} = \sigma^{(n)} \) of the image of \( \tilde{\sigma}^{(n)} \) under the transformation: \( \zeta = 1/w \).

Let \( l_1, l_2, \ldots, l_n \) be measures defined on \( \tilde{\zeta}^{(n)} \) as in definition (2.6); let \( h_1, h_2, \ldots, h_n \) be measures defined on \( \tilde{\sigma}^{(n)} \) in the same way. Since \( d(E_{\zeta}) = 1 \) it follows by Theorem (2.9) that: \( \prod_{k=1}^{n} l_k \leq 4 \). The points that contribute to the measure \( l_{n-k+1} \) are points in the complement of the image of the set of points contributing to \( h_k \) under \( \zeta = 1/w \). For fixed \( h_k \), the measure \( l_{n-k+1} \) is minimized when the set whose measure is \( h_k \) is the segment \([0, h_k]\) in which case: \( l_{n-k+1} = 1/h_k \). Thus:

\[
\prod_{k=1}^{n} l_k \geq \prod_{k=1}^{n} \frac{1}{h_k}
\]

and so:

\[
4 \geq \prod_{k=1}^{n} \frac{1}{h_k} \quad \text{or: } \left( \prod_{k=1}^{n} h_k \right)^{1/n} \geq \sqrt[4]{1/4}.
\]
Since the arithmetic mean exceeds the geometric mean:

$$\frac{1}{n} \sum_{k=1}^{n} h_k \geq \sqrt[4]{1/4}.$$ 

According to Remark (2.7): $\sum_{k=1}^{n} h_k = L$, the linear measure of $\sigma^*$. Thus: $L \geq n \cdot \sqrt[4]{1/4}$ as claimed.

**Theorem (3.2)** Let $w(z) \in S$ and $D_w$ the image of $|z| < 1$ under $w(z)$. Suppose $D_w \cap \{|w| = R\}$ consists of $n$ disjoint arcs $\{B_k\}_n$ where

(i) The angle subtended by the arc separating $B_k$ and $B_{k+1}$ is no greater than $2\pi/n$.

(ii) If $\{A_k^*\}_n$ are the $n$ arcs in the complement of $\bigcup_{k=1}^{n} B_k$ with respect to the circle $|w| = R$ the related set of arcs: $\{\eta_k \cdot A_k^*\}_n$ are nested.

Let the endpoints of the arc $B_k$ be given by: $R \cdot e^{i\theta_{2k-1}}$ and $R \cdot e^{i\theta_{2k}}$ ($k = 1, 2, \ldots, n$).

Then:

$$\prod_{k=1}^{n} \sin \left[ n\left(\theta_{2k+1} - \theta_{2k}\right)/4 \right] \leq R^{n^2}, \quad \theta_{2n+1} = \theta_1 + 2\pi.$$ 

**Proof.** Let $A_k^*$ be the arc lying between $B_k$ and $B_{k+1}$. The central angle subtended by $A_k^*$ is: $\theta_{2k+1} - \theta_{2k}$ which by hypothesis is no greater than $2\pi/n$. Let $A_k$ be the image of $A_k^*$ under the transformation $\zeta = 1/w$. The arcs $A_k$ all lie in the complement of $D_w$. Hence: $A = \bigcup_{k=1}^{n} A_k \subseteq E_\zeta$ and so $d(A) \leq d(E_\zeta) = 1$. The sets $A_k$ lie on the circle: $|\zeta| = 1/R$. The central angle subtended by $A_k$ is $\theta_{2k+1} - \theta_{2k}$; the same as that subtended by $A_k^*$. Finally, the arcs $A_k$ have the nested property hypothesized for the sets $A_k^*$. Since all this is so, Theorem (2.10) is applicable; therefore:

$$\prod_{k=1}^{n} \sin \left[ n\left(\theta_{2k+1} - \theta_{2k}\right)/4 \right] \leq \left[ (d(A)/(1/R))^{n^2} \right] = R^{n^2}$$

as claimed.

This past theorem takes no account of the fact that the complement of $D_w$ is a continuum containing the point at infinity. A sharpened version which takes this into account is the following:

$$d(0, 1, \theta_1 - \theta_2) \cdot \prod_{k=1}^{n} \sin \left[ n\left(\theta_{2k+1} - \theta_{2k}\right)/4 \right] \leq R^{n^2}$$

where $d(a, b, \theta)$ is as defined in §1. Actually, both Theorems (3.1) and (3.2) are generalized (in a sense, combined) in the following theorem, which takes the above fact into account. The techniques used to
prove the theorem are essentially the same as those of the foregoing proofs and so just a statement of the result will be given.

**Theorem (3.3).** Let \( f(z) \in S \) and \( D_w \) be the image of \( |z| < 1 \) under \( w = f(z) \). Let \( C \) be a circle of radius \( R \), \( 0 < R < \infty \) and \( n \) an arbitrary natural number. Let \( \{B_n\}_n \) be a sequence of arcs on the circle \( C \) satisfying the conditions of Theorem (3.2), \( S^{(n)} \) a set of \( n \)-fold symmetry generated by a broken ray and \( \tilde{S}^{(n)} \) a subset of \( S^{(n)} \) defined by: \( \tilde{S}^{(n)} = S^{(n)} \cap D_w \cap \{ |w| \leq R \} \). Let \( \sigma^{(n)} \) denote the circular projection of \( \tilde{S}^{(n)} \) and \( \{h_n\}_n \) a sequence of measures on \( \sigma^{(n)} \) such as defined in definition (2.6).

Then:

\[
d(0, \left[ \frac{R}{h_n} \right]^n, n[\theta_1 - \theta_2]) \cdot \prod_{k=2}^{n} d(1, \left[ \frac{R}{h_{n-k+1}} \right]^n, n[\theta_{2k+1} - \theta_{2k}]) \leq R^n.
\]

One final application will be given.

**Theorem (3.4).** Let \( f(z) \in S \) and \( D_w \) the image of the disk \( |z| < 1 \) under \( w = f(z) \). Let \( L_1, L_2 \) denote straight lines intersecting at \( w = 0 \) at an angle of \( \pi \alpha \), \( 0 < \alpha < 1 \). Let \( L = L(D_w \cap \{L_1 \cap L_2 \} \) denote the linear measure of \( D_w \cap \{L_1 \cup L_2 \} \). Then:

\[
L \geq \frac{2}{\alpha^{\alpha/2}(1 - \alpha)^{(1 - \alpha)/2}}.
\]

**Proof.** There is no loss in generality in assuming \( L_1 \) and \( L_2 \) are symmetric images of one-another with respect to the real axis.

A set of four points on the four legs determined by \( L_1 \cup L_2 \), each lying at a distance \( r \) from the origin, will be called a “radially symmetric set”; the points themselves will be called radially symmetric images of one-another and of the point \( w = 0 \).

We define \( h_k \) \((k = 1, 2, 3, 4)\) as the measure of the set of real numbers \( r \), \( 0 \leq r < \infty \) such that at least \( k \) of the radially symmetric images of \( r \) (in \( L_1 \cup L_2 \)) lie in \( D_w \). Then:

\[
(9) \quad L(D_w \cap \{L_1 \cup L_2 \}) = \sum_{k=1}^{4} h_k.
\]

Map by \( \zeta = 1/w \) and let \( E_\zeta \) represent the complement of the image of \( D_w \) under this map. Then \( d(E_\zeta) = 1 \). Notice that \( L_1 \cup L_2 \) is mapped onto itself. Let \( l_k \) be the measure of the set of real numbers \( r \) such that at least \( k \) of the radially symmetric images of \( r \) (in \( L_1 \cup L_2 \)) lie in \( E_\zeta \). Then:

\[
(10) \quad \prod_{k=1}^{4} l_k \geq \prod_{k=1}^{4} \frac{1}{h_k}. \]


Let \( T_1 = E_\zeta \cap \{L_1 \cup L_2\} \); let \( T_2 \) be the reflection of \( T_1 \) in the imaginary axis; let \( T_3 \) be the reflection of \( T_2 \) in the real axis; let \( T_4 \) be the reflection of \( T_3 \) in the imaginary axis. Clearly:

\[
d(T_1) = d(T_2) = d(T_3) = d(T_4).
\]

Let \( C_k \) be the set of all points contained in at least \( k \) of the \( T_j \)'s. The set \( C_k \) is a radially symmetric set; that is, it consists of all radially symmetric images of those points \( \zeta \) such that at least \( k \) of radially symmetric images of \( \zeta \) lie in \( T_1 \). Thus the measure of a leg of \( C_k \) is \( l_k \). Let \( B_k \) be the set consisting of four segments lying on the four rays determined by \( L_1 \cup L_2 \), each of length \( l_k \), the intersection of the four being the point \( \zeta = 0 \). Since the shift of segments that transforms \( C_k \) into \( B_k \) can only bring extremal points closer together, it follows that: \( d(C_k) \geq d(B_k) \). Using the mapping lemma (1.5) and Fekete's theorem (2.8) the transfinite diameter of \( B_k \) can be calculated:

\[
d(B_k) = \frac{l_k}{2\alpha^{n/2}(1 - \alpha)^{1-n/2}}.
\]

We have

\[
1 = d(E_\zeta) \geq d(T_1) \quad \text{since: } T_1 \subseteq E_\zeta
\]

\[
= \left[ \prod_{k=1}^{4} d(T_k) \right]^{1/4} \geq \left[ \prod_{k=1}^{4} d(C_k) \right]^{1/4} \quad \text{by Theorem (2.2)}
\]

\[
\geq \left[ \prod_{k=1}^{4} d(B_k) \right]^{1/4} = \left[ \prod_{k=1}^{4} \frac{l_k}{2\alpha^{n/2}(1 - \alpha)^{1-n/2}} \right]^{1/4}
\]

\[
\geq \frac{1}{2\alpha^{n/2}(1 - \alpha)^{1-n/2}} \left[ \prod_{k=1}^{4} \frac{1}{h_k} \right]^{1/4}
\]

\[
= \frac{1}{2\alpha^{n/2}(1 - \alpha)^{1-n/2}} \cdot \frac{4}{\sum_{k=1}^{4} h_k}
\]

since the arithmetic mean exceeds the geometric mean;

\[
= \left[ \frac{2}{(\alpha^{n/2}(1 - \alpha)^{1-n/2})} \right] (1/L).
\]

This sequence of inequalities means:

\[
L \geq \left[ \frac{2}{(\alpha^{n/2}(1 - \alpha)^{1-n/2})} \right].
\]

Remark. When \( \alpha = 1/2 \) that is, when \( L_1 \cup L_2 \) is a set of 4-fold symmetry, the result of the theorem reads: \( L \geq 2/(1/4)^{1/4} = 4(1/4)^{1/4} \) in agreement with Theorem (3.1).

I am grateful to the referee for supplying an abbreviated proof for Theorem (2.2).
REFERENCES


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