# Pacific Journal of Mathematics


## SOME TOPOLOGICAL PROPERTIES OF PIERCING POINTS


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Vol. 22, No. 2 February 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 2, 1967

# SOME TOPOLOGICAL PROPERTIES OF PIERCING POINTS

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**Let $K$ be the closure of one of the complementary domains of a 2-sphere $S$ topologically embedded in the 3-sphere, $S^3$. We give first (Theorem 1) a characterization of those points $p \in S$ with the following property: there exists a homeomorphism $h: K \to S^3$ such that $h(S)$ can be pierced with a tame arc at $h(p)$. The topological property of $K$ which distinguishes such a "piercing point" $p$ is this: $K - p$ is 1-ULC. Using this result, we find (Theorems 2 and 3) that $p$ is a piercing point of $K$ if and only if $S$ is arcwise accessible at $p$ by a tame arc from $S^3 - K$ (note: perhaps $S$ cannot be pierced with a tame arc at $p$, even if $p$ is a piercing point of $K$). Thus, the "tamely arcwise accessible" property is independent of the embedding of $K$ in $S^3$. The corollary to Theorem 2 gives an alternate proof of an as yet unpublished fact, first proven by R. H. Bing: a topological 2-sphere in $S^3$ is arcwise accessible at each point by a tame arc from at least one of its complementary domains.**

**In the last section of the paper, we give two applications of the above theorems. First, we show in Theorem 4 that $S$ can be pierced with a tame arc at $p$ if and only if $p$ is a piercing point of both $K$ and the closure of $S^3 - K$. Finally, we remark in Theorem 5 that S can be pierced with a tame arc at each of its points if it is "free" in the sense that for each $\varepsilon > 0$, $S$ can be mapped into each of its complementary domains by a mapping which moves each point less than $\varepsilon$. It is not known whether each 2-sphere $S$ with this last property is tame.**

A space homeomorphic to such a set $K$ in $S^3$ (as described at the beginning of the Introduction) is called a *crumpled cube*. We write $\mathrm{Bd}\, K = S$ and $\mathrm{Int}\, K = K - \mathrm{Bd}\, K$. An arc $A$ in $S^3$ is said to *pierce* a 2-sphere $S$ in $S^3$ if $A \cap S$ is an interior point $p$ of $A$ and the two components of $A - p$ lie in different components of $S^3 - S$. The *piercing points of a crumpled cube* are defined as above and were first considered by Martin [10]. It follows from Lemmas 2 and 3 of [10] and [6; Th. 11] that the nonpiercing points of a crumpled cube $K$ form a 0-dimensional $F_\sigma$ subset of Bd $K$.

If $C$ and $D$ are subsets of a space $Y$ with metric $d$, and $\varepsilon > 0$, we use $B(C, D; \varepsilon)$ to denote the set of all points $x \in D$ such that for some $y \in C$, $d(x, y) < \varepsilon$. The metric on $E^3$ and $S^3$ is always assumed to be the ordinary Euclidean one. Let $\Delta^n (n \geqq 1)$ denote a closed $n$-

simplex. If $Y$ is a metric space and $A \subset Y$, we say that $A$ is $n$-LC ($n \geqq 0$) at $p \in \mathrm{Cl}\, A \subset Y$ ($\mathrm{Cl}\, A =$ the closure of $A$) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each mapping of $\mathrm{Bd}\, \Delta^{n+1}$ into $B(p, A; \delta)$ extends to a mapping of $\Delta^{n+1}$ into $B(p, A; \varepsilon)$. We say that $A$ is $n$-ULC ($n \geqq 0$) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each mapping of $\mathrm{Bd}\, \Delta^{n+1}$ into a subset of $A$ of diameter less than $\delta$ extends to a mapping of $\Delta^{n+1}$ into a subset of $A$ of diameter less than $\varepsilon$. We refer to a mapping $f : \mathrm{Bd}\, \Delta^2 \to Y$ as a *loop*.

By a *null* sequence of subsets of a metric space, we mean one such that the diameters of its elements converge to zero. A *Sierpinski curve* $X$ is (uniquely) defined as any space homeomorphic to $[\mathrm{Bd}\, \Delta^3] - \bigcup \mathrm{Int}\, D_i$, where $D_1, D_2, \cdots$, is a null sequence of disjoint 2-cells whose union is a dense subset of $\mathrm{Bd}\, \Delta^3$. The *inaccessible part* of $X$ corresponds to $[\mathrm{Bd}\, \Delta^3] - \bigcup \mathrm{D}_i$. For a more detailed discussion of Sierpinski curves, see [3].

**2. Preliminary lemmas.** The following is Theorem 1 of [12], stated here for the reader's convenience.

LEMMA 1. *Let $C$ be a $q$-cell ($q = 1, 2,$ or 3) topologically embedded in $E^3$, and let $D \subset \mathrm{Bd}\ C$ be a ($q - 1$)-cell. Let $A_1, A_2, \cdots, A_k$ be a finite disjoint collection of tame arcs in $E^3 - D$ with each $\mathrm{Bd}\ A_i \subset E^3 - C$. Then, there exists a compact set $E \subset C - D$ such that, for each $\varepsilon > 0$, there is a homeomorphism $h : E^3 \to E^3$ with each $h(A_i) \subset E^3 - C$ and $h$ is the identity outside the $\varepsilon$-neighborhood of $E$.*

We shall also need the following [5; Th. 2].

LEMMA 2. *Let $B$ be a closed subset of $\Delta^2$; let $A$ be a subset of the separable metric space $Y$ and suppose that $A$ is 0-LC and 1-LC at each point of $Y$. Let $\varepsilon > 0$ and a mapping $f : \Delta^2 \to \mathrm{Cl}\ A$ be given. Then, There is a mapping $f^* : \Delta^2 \to \mathrm{Cl}\ A$ such that*

$$f^*(\Delta^2 - B) \subset A, f^* \mid B = f \mid B, \text{ and } d(f^*(x), f(x)) < \varepsilon$$

*for each $x \in \Delta^2$, where $d$ is the metric for $Y$.*

Let $X$ be a topological space, and $Y$ a closed subset of $X$. A loop $f : \mathrm{Bd}\, \Delta^2 \to X$ will be said to be *contractible in $X$ (mod $Y$)* if there exists a connected open set $N$ in $\Delta^2$ such that $\mathrm{Bd}\, \Delta^2 \subset N$, and a mapping $F : \mathrm{Cl}\, N \to X$ such that $F \mid \mathrm{Bd}\, \Delta^2 = f$, and $F$ maps the (point-set) boundary of $N$ (in $\Delta^2$) into $Y$.

LEMMA 3. *Let $K$ be a crumpled cube in $S^3$, and let $U$ be an* *open subset of $K$ such that $U \cap \operatorname{Bd} K$ is an open 2-cell $T$. Let $A$ be a compact subset of $K$ such that $A \cap \operatorname{Bd} K$ consists of a single point $p$ in $T$, where $K^* - p$ is 1-LC at $p$ and $K^*$ is the crumpled cube $S^3 - \operatorname{Int} K$. Then, if a loop in $U - A$ is contractible in $U - A$ (mod $T - p$), it is contractible to a point in $(U - A) \cup (W - A)$, where $W$ is any open set in $S^3$ containing $p$.*

*Proof.* Let $N$ be a connected open set in $\Delta^2$ containing $\operatorname{Bd} \Delta^2$, let $W$ be an open set in $S^3$ containing $p$, and let

$$F : \operatorname{Cl} N \rightarrow U - A$$

be a mapping which takes the boundary $B$ of $N$ in $\Delta^2$ into $T - p$. By the homotopy extension theorem, $F \mid B : B \rightarrow T$ extends to a mapping $G : \Delta^2 \rightarrow T$. Hence, by Lemma 2, and the fact that $K^* - p$ is 1-LC at $p$, $F \mid B$ extends to

$$G^* : \Delta^2 \rightarrow [T - p] \cup [(W \cap K^*) - p] .$$

Finally, define $H : \Delta^2 \rightarrow (U - A) \cup (W - A)$ by $H \mid \operatorname{Cl} N = F \mid \operatorname{Cl} N$ and $H \mid \Delta^2 - N = G^* \mid \Delta^2 - N$. Then $H$ is the required contraction of $F \mid \operatorname{Bd} \Delta^2$.

REMARK. Given the notation of the lemma, and a loop $f : \operatorname{Bd} \Delta^2 \rightarrow U - A$, a necessary condition for $f$ to be contractible to a point in $(U - A) \cup (W - A)$, where $W$ is a small neighborhood of $p$ in $S^3$, is that $f$ be contractible in $U - A$ (mod $T - p$).

## 3. Characterizations of piercing points.

THEOREM 1. *Let $K$ be a crumpled cube and $p$ a point of $\operatorname{Bd} K$. Then $p$ is a piercing point of $K$ if and only if $K - p$ is 1-LC at $p$.*

*Proof.* We may assume, by [8] and [9], that $K$ is embedded in $S^3$ in such a manner that there exists a homeomorphism $h$ of $C$, the closure of $S^3 - K$, onto the closed unit ball in $E^3$. Let $A$ be the inverse image under $h$ of the straight line segment in $E^3$ from the origin to $h(p)$. Then $A$ is an arc which is locally tame in $S^3$ except possibly at $p$, and according to Martin [10], $p$ is a piercing point of $K$ if and only if $A$ is tame. By [11, Lemma 5], $A$ is tame if and only if $S^3 - A$ is 1-LC at p. Hence the problem is reduced to showing that $S^3 - A$ is 1-LC at $p$ if and only if $K - p$ is 1-LC at $p$.

We shall give the details of the "if" part of the above assertion. The converse is merely a rearrangement of the same ideas. Suppose

$K - p$ is 1-LC at $p$, and let $\varepsilon$ be a positive number. We must find a $\delta > 0$ such that each loop in $B(p, S^3 - A; \delta)$ is contractible in $B(p, S^3 - A; \varepsilon)$. We assume that $\varepsilon$ is less than the distance from $p$ to $h^{-1}((0, 0, 0))$. Since $K - p$ is 1-LC at $p$, there exists $\rho > 0$ such that each loop in $B(p, K - p; \rho)$ is contractible in $B(p, K - p; \varepsilon)$. Let $U$ be an open subset of $S^3$ such that $p \in U \subset B(p, S^3; \rho)$ and such that there is a homeomorphism of $U \cap C$ onto the set of points in $E^3$ having nonnegative $z$-coordinates which takes $U \cap A$ into the $z$-axis. Finally, choose $\delta > 0$ so that $B(p, S^3; \delta) \subset U$.

Now, a given loop in $B(p, S^3 - A; \delta)$ is homotopic in $U - A$ to a loop in

$$(U \cap K) - p \subset B(p, K - p; \rho) ,$$

and this loop in turn is contractible to a point in $B(p, K - p; \varepsilon)$, as required.

REMARK. Since $K$ is compact and locally contractible, the condition "$K - p$ is 1-LC at $p$" is equivalent to "$K - p$ is 1-ULC".

COROLLARY. *Let $K$ be a crumpled cube, and $p$ a point of $S = \operatorname{Bd} K$. Then $p$ is a piercing point of $K$ if and only if the following condition holds: For each $\varepsilon > 0$, there is a $\delta > 0$ such that each simple closed curve in $B(p, S - p; \delta)$ is contractible in $B(p, K - p; \varepsilon)$.*

*Proof.* The condition is necessary by the preceding theorem. To show sufficiency, assume the notation of the preceding proof and let $\varepsilon > 0$ be given as before. Let $\delta > 0$ be chosen to satisfy the above condition and so that only the component of $A - B(p, S^3; \delta)$ which contains $h^{-1}((0, 0, 0))$ fails to lie in $B(p, S^3; \varepsilon)$. We also assume that $A$ is locally polyhedral at each point of $A - p$. Then, each piecewise-linear homeomorphism

$$f : \operatorname{Bd} \Delta^2 \to B(p, S^3 - A; \delta)$$

extends to a piecewise-linear mapping $F$ of $\Delta^2$ into $B(p, S^3 - p; \delta)$ such that $F$ is in general position relative to $A$. Hence $F^{-1}(A)$ is finite. If $x \in F^{-1}(A)$, then $F$ restricted to a sufficiently small curve enclosing $x$ represents a loop in $B(p, S^3 - A; \delta)$ which is homotopic in $B(p, S^3 - A; \varepsilon)$ to a loop in $B(p, S - p; \delta)$, and hence is contractible in $B(p, K - p; \varepsilon)$. This permits us to redefine $F$ in a small neighborhood of each $x \in F^{-1}(A)$, and thus obtain an extension of $f$ mapping $\Delta^2$ into $B(p, S^3 - A; \varepsilon)$. Hence $S^3 - A$ is 1-LC at $p$ and the result follows.

LEMMA 4. *Let $K$ be a crumpled cube in $S^3$, and $p$ a piercing* *point of the crumpled cube* $K^* = S^3 - \operatorname{Int} K$. *Suppose* $A$ *is an arc in* $K$ *having* $p$ *as an end-point, such that* $A \cap S = p$, *where* $S = \operatorname{Bd} K$. *If there exists a homeomorphism* $h: K \to S^3$ *such that* $h(A)$ *is tame, then* $A$ *is tame.*

*Proof.* Since $h(A)$ is tame, $A$ is locally tame in $S^3$ except possibly at $p$. Hence, by [11; Lemma 5], it suffices to show that $S^3 - A$ is 1-LC at $p$. Suppose $\varepsilon > 0$. Let $U$ be an open set in $S^3$ such that $p \in U \subset B(p, S^3; \varepsilon)$ and $U \cap S$ is an open 2-cell $T$. Since $h$ is a homeomorphism, and since $S^3 - h(A)$ is 1-LC at $h(p)$, there exists $\rho > 0$ such that each loop in $B(p, K - A; \rho)$ is contractible in $(U \cap K) - A$ (mod $T - p$). Choose $\mu > 0$ so that each loop in $B(p, K^*; \mu)$ is contractible in $B(p, K^*; \rho)$. Finally, let $\delta > 0$ be such that each pair of points in $B(p, S; \delta)$ can be joined by an arc in $B(p, S; \mu)$.

Now let a loop in $B(p, S^3 - A; \delta)$ be given. We give here an outline of the proof that this loop is contractible in $B(p, S^3 - A; \varepsilon)$. The details are left to the reader. There are three steps:

1. After performing a small homotopy in $B(p, S^3 - A; \delta)$, we assume that this loop is a simple closed curve $J$ such that $J \cap K^*$ consists of a finite number of disjoint arcs $L_1, L_2, \cdots, L_k$, with $L_i \cap S = \operatorname{Bd} L_i$, for each $i$.

2. For each $i$, let $Z_i$ be an arc in $B(p, S; \mu) - p$ joining the end-points of $L_i$. Then $L_i$ is homotopic in $B(p, K^*; \rho)$, with end-points fixed, to $Z_i$. Since $K^* - p$ is 1-LC at $p$, Lemma 2 allows us to adjust this homotopy to give one in $B(p, K^*; \rho) - p$ between $L_i$ and $Z_i$. Hence, by piecing together these homotopies, we see that the given loop is homotopic in $B(p, S^3 - A; \rho)$ to the loop

$$[J - \bigcup \operatorname{Int} L_i] \cup \bigcup Z_i$$

in $B(p, K - A; \rho)$.

3. This last loop is contractible in $(U \cap K) - A$ (mod $T - p$). Hence, by Lemma 3, it is contractible to a point in $B(p, S^3 - A; \varepsilon)$. This completes the proof.

REMARK. Using the same techniques, and Lemma 3, we could prove this lemma with "tame" replaced consistently by "cellular" or "has a simply-connected complement in $S^3$" everywhere in its statement. In these two alternate formulations, we could permit $A$ to be any compact absolute retract, and $p$ any point of $A$.

THEOREM 2. *Let* $K$ *be a crumpled cube in* $S^3$, *and* $p$ *a point of* $S = \operatorname{Bd} K$. *If* $p$ *is a piercing point of* $K$, *then there is a tame arc* $A$ *in* $K^* = S^3 - \operatorname{Int} K$ *having* $p$ *as an end-point such that* $A \cap S = p$.

*Proof.* By Lemma 4, it suffices to show that there is an arc $A$ in $K^*$ having $p$ as an end-point such that $A \cap S = p$, and such that for some embedding $h : K^* \to S^3$, $h(A)$ is tame. We choose $h$ so that the closure of $S^3 - h(K^*)$ is a 3-cell ([8] and [9]). Hence, the theorem will follow as stated above if we can prove it in the special case when $K$ is a closed 3-cell. We make this assumption to simplify the notation.

Let $f$ be a homeomorphism of the closed unit ball $B$ in $E^3$ onto $K$, with $f((0, 0, 1)) = p$. Let $T_i\,(i = 1, 2, \cdots)$ be the 2-cell which is the $f$-image of the intersection of $B$ with the plane $z = 1 - 1/i$. Let the 3-cell $C_i\,(i = 1, 2, \cdots)$ be defined inductively as follows: $C_1$ is the closure of the component of $K - T_1$ not containing $p$; $C_i\,(i \geqq 2)$ is the closure of the component of

$$K - T_i - \bigcup_{j<i} C_j$$

not containing $p$. Finally, let $A^*$ be a tame arc in $S^3$ having $p$ as one end-point and the other end-point not in $K$. We assume that $A^* \cap C_1 = \phi$.

According to Lemma 1, there is for each $i > 1$, a homeomorphism $g_i : S^3 \to S^3$ which is the identity outside a small neighborhood $U_i$ of $T_i$ and which is such that $g_i(A^*) \cap T_i = \phi$. In particular, the $U_i$'s may be chosen to form a null sequence of disjoint sets. Let $g$ be the homeomorphism of $S^3$ onto itself which agrees with $g_i$ on $U_i$, for each $i$, and otherwise is the identity. Then $g(A^*) \cap T_i = \phi$, for each $i$, and $g(p) = p$.

Again using Lemma 1, there is, for each $i > 1$, a compact set $E_i \subset C_i - (T_i \cup T_{i-1})$ (by the previous paragraph, there is a 2-cell in $\operatorname{Bd} C_i$ containing $T_i \cup T_{i-1}$ and missing $g(A^*)$) and a homeomorphism $k_i : S^3 \to S^3$ which is the identity outside an arbitrarily small neighborhood $V_i$ of $E_i$ and which is such that $k_i g(A^*) \cap C_i = \phi$, for each $i$. We choose $V_i$ so close to $E_i$ that the $V_i$'s form a null sequence of disjoint sets, and so that $V_i$ misses the closure of $K - C_i$. Let $k$ be the homeomorphism of $S^3$ onto itself which agrees with $k_i$ on $V_i$, for each $i$, and reduces to the identity otherwise. Then $A = kg(A^*)$ is the required arc.

COROLLARY (*Bing*). *A topological 2-sphere in $S^3$ is arcwise accessible at each point by a tame arc from at least one of its complementary domains.*

*Proof.* Let $K$ and $K^*$ be the two crumpled cubes into which the 2-sphere $S$ decomposes $S^3$. If $p \in S$, then either $p$ is a piercing point of $K$, or $p$ is a piercing point of $K^*$([10; Theorem]). The result then follows from the preceding theorem.

THEOREM 3. *Let $K$ be a crumpled cube in $S^3$, and $p$ a point of $S = \operatorname{Bd} K$. If there is a tame arc $A$ in $K^* = S^3 - \operatorname{Int} K$ having $p$ as an end-point and such that $A \cap S = p$, then $p$ is a piercing point of $K$.*

*Proof.* It suffices to establish the condition given in the corollary to Theorem 1. Thus, take $\varepsilon > 0$. We assume that $\varepsilon$ is less than the distance between $p$ and $q$, where $q$ is the other end-point of $A$. Choose $\delta > 0$ so that $B(p, S; \delta)$ lies interior to a closed 2-cell $D \subset B(p, S; \varepsilon)$.

Since $A$ is locally tame at $p$, there is a tame 2-sphere

$$Z^* \subset B(p, S^3; \delta)$$

which separates $p$ from $q$ in $S^3$ and which meets $A$ at precisely one point $r \in \operatorname{Int} A$, at which $A$ pierces $Z^*$. Let $T$ be a small closed 2-cell in $Z^*$ missing $K$ and such that $r \in \operatorname{Int} T$. Note that, by linking considerations, $\operatorname{Bd} T$ is not contractible in $B(p, K^*; \varepsilon) - A$.

Appealing to [2; Th. 1] and [4; Th. 1], we obtain, for each $\rho > 0$, a tame Sierpinski curve $X \subset S$ such that each component $U_i$ $(i = 1, 2, \cdots)$ of $S - X$ has diameter less than $\rho$, and a homeomorphism $h : S^3 \to S^3$ which moves each point of $S^3$ less than $\rho$, which is the identity outside $B(Z^* \cap S, S^3; \rho)$, and which is such that $h(Z^*) \cap X$ consists of a finite disjoint collection of simple closed curves each in the inaccessible part of $X$. Let $Z = h(Z^*)$. By choosing $\rho$ sufficiently small, we may ensure that $h$ is the identity on $T$ and that $Z$ retains all the properties originally required of $Z^*$. A final requirement on $\rho$ is that $\rho < \varepsilon - \delta$ and that the component of $S - X$ containing $p$ should not meet $Z$ (if $p \in X$, then $S$ can be pierced with a tame arc at $p$, by [6; Th. 6]).

We assert that there is at least one component of $Z \cap S$ separating $p$ from $\operatorname{Bd} D$ in $D$ (this component is necessarily a simple closed curve). If not, then $Z \cap X$ consists of a finite number of simple closed curves each of which is contractible in $D - p$, and $Z \cap (S - X)$ can be covered by the null sequence of disjoint open 2-cells of diameter less than $\rho$ in $S : U_1, U_2, \cdots$. Note that $U_i \cap Z$ is compact. It is now easy, using the homotopy extension theorem on each of the inclusions $U_i \cap Z \to U_i$ as in the proof of Lemma 3, to construct a mapping contracting $\operatorname{Bd} T$ in

$$[K^* \cap (Z - \operatorname{Int} T)] \cup [B(p, S - p; \varepsilon)] \subset B(p, K^*; \varepsilon) - A ,$$

a contradiction.

By the preceding paragraph, we may let $L$ be an innermost (in $Z - T$) one of the components of $S \cap Z$ which separates $p$ from Bd $D$ in $D$. Let $L$ bound the 2-cell $F \subset Z - T$. Note that $L$ is not contractible in $B(p, K^*; \varepsilon) - A$ and that no component of $S \cap \operatorname{Int} F$ separates $p$ from Bd $D$ in $D$. Hence, by the argument of the preceding paragraph, the "large" component of $F - S$ lies in Int $K$, and $L$ is contractible in

$$[K \cap F] \cup [B(p, S - p; \varepsilon)] \subset B(p, K - p; \varepsilon) .$$

Since each simple closed curve in $B(p, S - p; \delta)$ is homotopic in $D - p$ to $L$, the proof is complete.

## 4. Some applications.

THEOREM 4. *Let $S$ be a 2-sphere topologically embedded in $S^3$, and let $K$ and $K^*$ be the two crumpled cubes into which $S$ divides $S^3$. Then $S$ can be pierced with a tame arc at a point $p \in S$ if and only if $p$ is a piercing point of $K$ and a piercing point of $K^*$.*

*Proof.* The "only if" part of the theorem follows from Theorem 3. For the converse, suppose that $p$ is a piercing point of each of $K$ and $K^*$, and let $A$ be an arc in $S$ such that $A$ is locally tame except possibly at the end-point $p$. By [6; Th. 6], $S$ can be pierced with a tame arc at $p$ if $A$ is tame.

To show that $A$ is tame, we proceed in essentially the same manner as in the proof of [6; Lemma 6.1]. That is, let $S'$ be a 2-sphere in $S^3$ which contains $A$ and is locally tame at each point of $S' - A$, and which is homeomorphically so close to $S$ that $p$ is a piercing point of each of the crumpled cubes $L$ and $L^*$ into which $S'$ divides $S^3$ (use Theorems 2 and 3). It suffices to show that $S'$ is tame.

Exactly as in [6], $S'$ is locally tame at each point of $A - p$. Hence, $S'$ is locally tame except possibly at $p$. It follows easily, since $L - p$ and $L^* - p$ are each 1-LC at $p$, that $S^3 - S'$ is 1-LC at each point of $S'$ and hence that $S'$ is tame by [1; Th. 6]. This completes the proof.

In [7], Hempel studied the properties of a surface $S$ ($=$Bd $K$) which is *free* relative to one of its complementary domains (Int $K$) in $S^3$ (i.e., $S$ satisfies the mapping condition stated in the following theorem). It is not known whether the crumpled cube of this theorem is necessarily a 3-cell.

THEOREM 5. *Let $K$ be a crumpled cube, and let $S =$ Bd $K$. Suppose that for each $\varepsilon > 0$ there exists a mapping $f: S \to$ Int $K$ which* *moves each point of $S$ less than $\varepsilon$. Then each point of $S$ is a piercing point of $K$.*

*Proof.* We shall verify the condition given in the corollary to Theorem 1. Suppose $p \in S$ and $\varepsilon > 0$. Choose $\delta > 0$ so that there is a closed 2-cell $D \subset S$ such that

$$B(p, S; \delta) \subset D \subset B(p, S; \varepsilon) .$$

Then, if $J$ is a simple closed curve in $B(p, S - p; \delta)$ bounding a 2-cell $D^* \subset D$, there is a $\rho > 0$ such that $\rho$ is less than the distance from $D$ to the complement of $B(p, K; \varepsilon)$ and such that each mapping of $J$ into $K$ which moves each point of $J$ less than $\rho$ is homotopic in $B(p, K - p; \delta)$ to the inclusion of $J$ into $B(p, K - p; \delta)$.

Suppose $f: S \to \mathrm{Int}\, K$ is a mapping which moves each point of $S$ less than $\rho$. Then $J$ is homotopic in $B(p, K - p; \delta)$ to $f(J)$, and $f(J)$ bounds the singular 2-cell

$$f(D^*) \subset B(p, K; \varepsilon) - S .$$

This completes the proof.

REMARK. If $S \subset S^3$ is a topological 2-sphere which is free relative to *each* of its complementary domains, then it follows from the foregoing theorems that $S$ can be pierced with a tame arc at each of its points.

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Received July 25, 1966. This research was supported in part by Grant NSF GP-4125. The author is an Alfred P. Sloan Fellow.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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