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**In this paper we show that the 1-section of a finite simplicial complex M is characterized by the topological type of the 1-section of the cone over M. This enables us to prove that a finite simplicial complex is characterized by the topological type of the 1-section of the first derived complex of its cone.**

R. L. Finney [1] proved that two locally-finite simplicial complexes are isomorphic if their first derived complexes are isomorphic. J. Segal [3] making use of this showed that two locally-finite simplicial complexes are isomorphic if the 1-sections of their first derived complexes are isomorphic. He then showed in [4] that a restricted class of finite complexes are characterized by the topological type of the 1-section of their first derived complexes. In contrast to [4] the results of this paper apply without restricting the class of finite complexes.

Throughout, $s_p$ will denote a (rectilinear) $p$-simplex; $M$ will denote a finite geometric simplicial complex with $r$-section $M^r$ and first derived complex $M'$. The cone at $m$ over $M$, $m \notin M$, is denoted by $mM$. For more details see [2, 1.2]. We only consider complexes with at least two vertices.

LEMMA. (a) *If $mM$ and $nN$ are isomorphic then so are $M$ and $N$.* (b) *If $(mM)^1$ and $(nN)^1$ are isomorphic then so are $M^1$ and $N^1$.*

*Proof.* (a) Let $\varphi$ be an isomorphism of $mM$ onto $nN$. If $\varphi(m) = n$ we are done so we assume $\varphi(m) \neq n$, hence we also have $\varphi^{-1}(n) \neq m$.

Given a complex $K$, with vertex $v$, the subcomplex consisting of those simplexes not having $v$ as a vertex is denoted $K\langle v\rangle$.

We now define subcomplexes of $M$ and $N$ as follows:

$$M_1 = (mM)\langle\varphi^{-1}(n)\rangle \quad , \; M_2 = M\langle\varphi^{-1}(n)\rangle$$
$$N_1 = (nN)\langle\varphi(m)\rangle \quad , \; N_2 = N\langle\varphi(m)\rangle \, .$$

The following relationships are easily verified:

(1) $M_1 = mM_2, N_1 = nN_2$ ;

(2) $\varphi | M_2$ is an isomorphism of $M_2$ onto $N_2$ ;

(3) $\varphi | M$ is an isomorphism of $M$ onto $N_1$, and $\varphi^{-1\prime}N$ is an isomorphism of $N$ onto $M_1$.

Using $\approx$ to denote isomorphism we then have:

$$M \approx N_1 = nN_2 \approx mM_2 = M_1 \approx N .$$

Here the first and last isomorphisms follow from (3), the equalities from (1) and the middle isomorphism follows from (2) and the fact that taking cones preserves isomorphism.

A proof of part (b) is obtained by taking 1-sections at appropriate places in the above argument.

THEOREM 1. *If* $|(mM)^1|$ *and* $|(nN)^1|$ *are homeomorphic then* $M^1$ *and* $N^1$ *are isomorphic.*

*Proof.* For nontriviality we assume each of $M$ and $N$ has at least 3 vertices. Let $T_M$ denote the set of vertices of $mM$ whose order in $|(mM)^1|$ is not 2 and let $T_N$ be the corresponding set in $nN$; then $m$ is in $T_M$ and $n$ is in $T_N$.

Now let $h$ be a homeomorphism of $|(mM)^1|$ onto $|(nN)^1|$. We shall modify $h$ where necessary to get a new homeomorphism $\tilde{h}$ which maps the vertices of $(mM)^1$ onto those of $(nN)^1$.

Clearly any homeomorphism of $|(mM)^1|$ onto $|(nN)^1|$ takes $T_M$ onto $T_N$. Let $v_1, \cdots, v_r$ be the vertices of $(mM)^1$ having order 2; we show how to construct homeomorphisms $h_1, \cdots, h_r$ such that

$$h_i\colon |(mM)^1| \longrightarrow |(nN)^1|$$

and for $i \leqq j$, $h_j(v_i)$ is a vertex of $(nN)^1$. Starting with $h$ we shall construct $h_1$; the construction of $h_i$ from $h_{i-1}$ follows the same pattern and will be omitted.

For simplicity we write $v$ rather than $v_1$. If $h(v)$ is a vertex of $(nN)^1$ we let $h_1 = h$. Suppose then that $h(v)$ is not a vertex. Let $C$ be the closure in $|(mM)^1|$ of the component $Q$ of $|(mM)^1| - T_M$ containing $v$; then $C$ is either an arc with endpoints in $T_M$ or a simple closed curve which one easily shows must be of the form $Q \cup \{m\}$.

Suppose first it is an arc with endpoints $x$ and $y$. Using the fact that $v$ has order 2 in $|(mM)^1|$ we conclude that one of $x, y$, say $x$, is $m$ and that $Q$ contains no vertex of $mM$ other than $v$.

Let $\sigma$ be the 1-simplex of $mM$ spanned by $m$ and $y$. Applying $h$, we get a pair of arcs $h(|\sigma|)$ and $h(C)$ in $|(nN)^1|$ whose union is a simple closed curve containing exactly two points of $T_N$-namely $h(m)$ and $h(y)$. It follows that there is a vertex $w$ of $nN$ which lies either on $h(|\sigma|\text{-}\{m, y\})$ or $h(C - \{m, y\})$. In the first case we choose a self-homeomorphism $k$ of $|(nN)^1|$ which is the identity off $h(|\sigma| \cup C)$, interchanges $h(|\sigma|)$ and $h(C)$ leaving $h(m)$ and $h(y)$ fixed, and takes $h(v)$ onto $w$; we define $h_1 = k \circ h$. The second case is similar-except that $k$ is taken as the identity off $h(Q)$.

If $C$ is a simple closed curve, $C = Q \cup \{m\}$, then $Q$ must contain exactly two vertices of order 2, say $v$ and $w$. Since $h(C) \cap T_N = \{h(m)\}$ it follows that $h(m) = n$ and $h(Q)$ contains exactly two vertices of order 2, say $v'$ and $w'$. In this case we choose a self-homeomorphism $k$ of $|(nN)^1|$ which is the identity off $h(Q)$ and takes $h(v)$ to $v'$ and $h(w)$ to $w'$. The composition $h_1 = k \circ h$ has the desired properties. This completes the construction of $h_1$.

We let $\tilde{h} = h_r$; then $\tilde{h}$ takes each vertex of $(mM)^1$ to a vertex of $(nN)^1$. In particular $(mM)^1$ has at least as many vertices as $(nN)^1$. Since a similar construction can be made starting with $h^{-1}$, the number of vertices in each complex is the same. Hence the homeomorphism $\tilde{h}$ takes the vertices of $(mM)^1$ onto those of $(nN)^1$. It follows (see, for example, the argument of Theorem 3 of [4]) that $\tilde{h}$ induces an isomorphism of $(mM)^1$ onto $(nN)^1$.

Applying part (b) of the lemma, we have that $M^1$ and $N^1$ are isomorphic.

DEFINITION. An $n$-complex $M$ is *full* provided, for any subcomplex $K$ of $M$ which is isomorphic to $s_p^1$, $2 \leqq p \leqq n$, $K^0$ spans a $p$-simplex of $M$.

THEOREM 2. *If $M$ and $N$ are full complexes, then they are isomorphic if $|(mM)^1|$ and $|(nN)^1|$ are homeomorphic.*

This follows from Theorem 1 and Theorem 1 of [3] which says that if $M$ and $N$ are full and $M^1$ and $N^1$ are isomorphic, then $M$ and $N$ are isomorphic.

DEFINITION. Given the cone at $m$ over $M$ and a subcomplex $K$ of $M$ we define the *tower-complex* over $K$ (relative to $mM$) to be $((mK)')^1$ and we denote it by $t_m(K)$. Furthermore, we call the underlying polyhedron of $t_m(K)$ the *tower* over $K$ (relative to $mM$) and denote it by $t(K)$, i.e.., $t(K) = |t_m(K)|$.

THEOREM 3. *If $M$ and $N$ are complexes, then $M$ and $N$ are isomorphic if and only if $t(M)$ and $t(N)$ are homeomorphic.*

*Proof.* Suppose $t(M)$ and $t(N)$ are homeomorphic. We first assume that $M$ and $N$ have no vertices of order 0. Then the order of each vertex of $(mM)'$ in $t_m(M)$ and of $(nN)'$ in $t_m(N)$ is at least three. So we may apply Theorem 5 of [4] to obtain an isomorphism between $mM$ and $nN$. This by part (a) of the Lemma yields the desired isomorphism between $M$ and $N$.

Now consider the case in which $M$ or $N$ has vertices of order 0. Let $K$ denote the set of vertices of $M$ which are of order 0 and let

$L$ be the corresponding set for $N$. Let $\tilde{M} = M - K$ and $\tilde{N} = N - L$. Then

$$t(M) = t(\tilde{M}) \cup t(K)$$

and

$$t(N) = t(\tilde{N}) \cup t(L) .$$

Let $h$ be a homeomorphism of $t(M)$ onto $t(N)$. Since $t(K)$ is the smallest connected subset of $t(M)$ that contains $K$, the set $h(t(K))$ is the smallest connected subset of $t(N)$ that contains $h(K)$. But $h(K) = L$, because the points of $K$ and $L$ are the only ones with order 1 in $t(M)$ and $t(N)$. Therefore, $h(t(K)) = t(L)$, and by taking complements we see that $h(t(\tilde{M})) = t(\tilde{N})$. Therefore, by the preceding argument, there exists an isomorphism $f$ of $\tilde{M}$ onto $\tilde{N}$. Since $h$ yields an isomorphism of $K$ and $L$, $f$ can be extended to an isomorphism of $M$ and $N$.

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