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ISOMORPHIC CONE-COMPLEXES

JACK SEGAL AND EDWARD SANDUSKY THOMAS, JR.

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In this paper we show that the 1-section of a finite simplicial complex M is characterized by the topological type of the 1-section of the cone over M . This enables us to prove that a finite simplicial complex is characterized by the topological type of the 1-section of the first derived complex of its cone.

R. L. Finney [1] proved that two locally-finite simplicial complexes are isomorphic if their first derived complexes are isomorphic. J. Segal [3] making use of this showed that two locally-finite simplicial complexes are isomorphic if the 1-sections of their first derived complexes are isomorphic. He then showed in [4] that a restricted class of finite complexes are characterized by the topological type of the 1-section of their first derived complexes. In contrast to [4] the results of this paper apply without restricting the class of finite complexes.

Throughout, s_p will denote a (rectilinear) p -simplex; M will denote a finite geometric simplicial complex with r -section M^r and first derived complex M' . The cone at m over M , $m \in M$, is denoted by mM . For more details see [2, 1.2]. We only consider complexes with at least two vertices.

LEMMA. (a) *If mM and nN are isomorphic then so are M and N .* (b) *If $(mM)^1$ and $(nN)^1$ are isomorphic then so are M^1 and N^1 .*

Proof. (a) Let φ be an isomorphism of mM onto nN . If $\varphi(m) = n$ we are done so we assume $\varphi(m) \neq n$, hence we also have $\varphi^{-1}(n) \neq m$.

Given a complex K , with vertex v , the subcomplex consisting of those simplexes not having v as a vertex is denoted $K\langle v \rangle$.

We now define subcomplexes of M and N as follows:

$$\begin{aligned} M_1 &= (mM)\langle \varphi^{-1}(n) \rangle, & M_2 &= M\langle \varphi^{-1}(n) \rangle \\ N_1 &= (nN)\langle \varphi(m) \rangle, & N_2 &= N\langle \varphi(m) \rangle. \end{aligned}$$

The following relationships are easily verified:

- (1) $M_1 = mM_2, N_1 = nN_2$;
- (2) $\varphi|_{M_2}$ is an isomorphism of M_2 onto N_2 ;
- (3) $\varphi|M$ is an isomorphism of M onto N_1 , and $\varphi^{-1}N$ is an isomorphism of N onto M_1 .

Using \approx to denote isomorphism we then have:

$$M \approx N_1 = nN_2 \approx mM_2 = M_1 \approx N.$$

Here the first and last isomorphisms follow from (3), the equalities from (1) and the middle isomorphism follows from (2) and the fact that taking cones preserves isomorphism.

A proof of part (b) is obtained by taking 1-sections at appropriate places in the above argument.

THEOREM 1. *If $|(mM)^1|$ and $|(nN)^1|$ are homeomorphic then M^1 and N^1 are isomorphic.*

Proof. For nontriviality we assume each of M and N has at least 3 vertices. Let T_M denote the set of vertices of mM whose order in $|(mM)^1|$ is not 2 and let T_N be the corresponding set in nN ; then m is in T_M and n is in T_N .

Now let h be a homeomorphism of $|(mM)^1|$ onto $|(nN)^1|$. We shall modify h where necessary to get a new homeomorphism \tilde{h} which maps the vertices of $(mM)^1$ onto those of $(nN)^1$.

Clearly any homeomorphism of $|(mM)^1|$ onto $|(nN)^1|$ takes T_M onto T_N . Let v_1, \dots, v_r be the vertices of $(mM)^1$ having order 2; we show how to construct homeomorphisms h_1, \dots, h_r such that

$$h_i: |(mM)^1| \rightarrow |(nN)^1|$$

and for $i \leq j$, $h_j(v_i)$ is a vertex of $(nN)^1$. Starting with h we shall construct h_1 ; the construction of h_i from h_{i-1} follows the same pattern and will be omitted.

For simplicity we write v rather than v_1 . If $h(v)$ is a vertex of $(nN)^1$ we let $h_1 = h$. Suppose then that $h(v)$ is not a vertex. Let C be the closure in $|(mM)^1|$ of the component Q of $|(mM)^1| - T_M$ containing v ; then C is either an arc with endpoints in T_M or a simple closed curve which one easily shows must be of the form $Q \cup \{m\}$.

Suppose first it is an arc with endpoints x and y . Using the fact that v has order 2 in $|(mM)^1|$ we conclude that one of x, y , say x , is m and that Q contains no vertex of mM other than v .

Let σ be the 1-simplex of mM spanned by m and y . Applying h , we get a pair of arcs $h(|\sigma|)$ and $h(C)$ in $|(nN)^1|$ whose union is a simple closed curve containing exactly two points of T_N -namely $h(m)$ and $h(y)$. It follows that there is a vertex w of nN which lies either on $h(|\sigma| - \{m, y\})$ or $h(C - \{m, y\})$. In the first case we choose a self-homeomorphism k of $|(nN)^1|$ which is the identity off $h(|\sigma| \cup C)$, interchanges $h(|\sigma|)$ and $h(C)$ leaving $h(m)$ and $h(y)$ fixed, and takes $h(v)$ onto w ; we define $h_1 = k \circ h$. The second case is similar-except that k is taken as the identity off $h(Q)$.

If C is a simple closed curve, $C = Q \cup \{m\}$, then Q must contain

exactly two vertices of order 2, say v and w . Since $h(C) \cap T_N = \{h(m)\}$ it follows that $h(m) = n$ and $h(Q)$ contains exactly two vertices of order 2, say v' and w' . In this case we choose a self-homeomorphism k of $|(nN)^1|$ which is the identity off $h(Q)$ and takes $h(v)$ to v' and $h(w)$ to w' . The composition $h_1 = k \circ h$ has the desired properties. This completes the construction of h_1 .

We let $\tilde{h} = h_r$; then \tilde{h} takes each vertex of $(mM)^1$ to a vertex of $(nN)^1$. In particular $(mM)^1$ has at least as many vertices as $(nN)^1$. Since a similar construction can be made starting with h^{-1} , the number of vertices in each complex is the same. Hence the homeomorphism \tilde{h} takes the vertices of $(mM)^1$ onto those of $(nN)^1$. It follows (see, for example, the argument of Theorem 3 of [4]) that \tilde{h} induces an isomorphism of $(mM)^1$ onto $(nN)^1$.

Applying part (b) of the lemma, we have that M^1 and N^1 are isomorphic.

DEFINITION. An n -complex M is *full* provided, for any subcomplex K of M which is isomorphic to s_p^1 , $2 \leq p \leq n$, K^0 spans a p -simplex of M .

THEOREM 2. *If M and N are full complexes, then they are isomorphic if $|(mM)^1|$ and $|(nN)^1|$ are homeomorphic.*

This follows from Theorem 1 and Theorem 1 of [3] which says that if M and N are full and M^1 and N^1 are isomorphic, then M and N are isomorphic.

DEFINITION. Given the cone at m over M and a subcomplex K of M we define the *tower-complex* over K (relative to mM) to be $((mK)')^1$ and we denote it by $t_m(K)$. Furthermore, we call the underlying polyhedron of $t_m(K)$ the *tower* over K (relative to mM) and denote it by $t(K)$, i.e., $t(K) = |t_m(K)|$.

THEOREM 3. *If M and N are complexes, then M and N are isomorphic if and only if $t(M)$ and $t(N)$ are homeomorphic.*

Proof. Suppose $t(M)$ and $t(N)$ are homeomorphic. We first assume that M and N have no vertices of order 0. Then the order of each vertex of $(mM)'$ in $t_m(M)$ and of $(nN)'$ in $t_m(N)$ is at least three. So we may apply Theorem 5 of [4] to obtain an isomorphism between mM and nN . This by part (a) of the Lemma yields the desired isomorphism between M and N .

Now consider the case in which M or N has vertices of order 0. Let K denote the set of vertices of M which are of order 0 and let

L be the corresponding set for N . Let $\tilde{M} = M - K$ and $\tilde{N} = N - L$. Then

$$t(M) = t(\tilde{M}) \cup t(K)$$

and

$$t(N) = t(\tilde{N}) \cup t(L).$$

Let h be a homeomorphism of $t(M)$ onto $t(N)$. Since $t(K)$ is the smallest connected subset of $t(M)$ that contains K , the set $h(t(K))$ is the smallest connected subset of $t(N)$ that contains $h(K)$. But $h(K) = L$, because the points of K and L are the only ones with order 1 in $t(M)$ and $t(N)$. Therefore, $h(t(K)) = t(L)$, and by taking complements we see that $h(t(\tilde{M})) = t(\tilde{N})$. Therefore, by the preceding argument, there exists an isomorphism f of \tilde{M} onto \tilde{N} . Since h yields an isomorphism of K and L , f can be extended to an isomorphism of M and N .

REFERENCES

1. R. L. Finney, *The insufficiency of barycentric subdivision*, Michigan Math. J. **12** (1965), 263-272.
2. P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge University Press, Cambridge, 1960.
3. J. Segal, *Isomorphic complexes*, Bull. Amer. Math. Soc. **71** (1965), 571-572.
4. ———, *Isomorphic complexes, II*, Bull. Amer. Math. Soc. **72** (1966), 300-302.

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