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This paper deals with (1) acceleration of the convergence of a convergent complex series, (2) rapidity of convergence, and (3) sufficient criteria for the divergence of a complex series. Various results of Samuel Lubkin, Imanuel Marx and J. P. King which concern or are closely related to Aitkin's δ^2-process are generalized. Some typical results are as follows:

(1) If a complex series and its δ^2-transform converge, their sums are equal.

(2) Suppose that $\Sigma a_n, \Sigma b_n$ are complex series such that $b_n/a_n \to 0$, and A, B exists such that $|a_n/a_{n-1}| \leqq A < 1/2$, $|b_n/b_{n-1}| \leqq B < 1$ for all sufficiently large n. Then Σb_n converges more rapidly than Σa_n.

(3) If the sequence $\{1/a_n - 1/a_{n-1}\}$ is bounded, then the complex series Σa_n diverges.

Given a convergent complex series $\Sigma a_n = S$, quantities $T_n = (a_n + a_{n+1} + \cdots)/a_{n-1}$ are used to obtain results on accelerating the convergence of Σa_n and on rapidity of convergence. The convergence of $\{T_n\}$ is treated and corresponding necessary and sufficient conditions are established for the transform $\Sigma a_{\alpha n} = S$ to converge more rapidly that Σa_n, where $a_{\alpha 0} = a_0 + a_1\alpha_1$, $a_{\alpha n} = a_n + a_{n+1}\alpha_{n+1} - a_n\alpha_n$ for $n \geqq 1$, and $\{\alpha_n\}$ is any complex sequence. Divergence theorems are proven, of which Theorem 2.8 furnishes a generalization of corrected results of Marx [10] and King [7]. The appropriate corrections are indicated in Tucker [16]. These divergence theorems are used to prove that if Σa_n and its δ^2-transform are convergent complex series, their sums are equal. This fact was first published by Lubkin [9] for real series. Theorem 2.9 gives a generalization of a theorem of Marx [10] and King [7], corrected statements of which are given in Tucker [16]. Some related theorems on rapidity of convergence are then proven. Before turning to the general analysis, we now present difinitions, notations and certain elementary facts relevant to acceleration.

Given a complex series $\sum_0^\infty a_n$, we shall write Σa_n for $\sum_0^\infty a_n$, $S_n = \sum_0^n a_k$, and, if Σa_n converges, $S = \Sigma a_n$. Similarly, if $\Sigma a'_n$ converges, then $S' = \Sigma a'_n$. Given two convergent series Σa_n and $\Sigma a'_n$, the latter is said to converge more rapidly than the former if and only if $(S' - S'_n)/(S - S_n) \to 0$ as $n \to \infty$. If Σa_n converges, "$MR(\Sigma a_n)$" will denote the class of all series Σb_n which converge more rapidly to S than Σa_n.

The concept of "acceleration" or "speed-up" can now be defined as the problem of finding a series Σb_n such that $\Sigma b_n \in MR(\Sigma a_n)$. We will say that $\Sigma a'_n$ converges with the same rapidity as Σa_n if and only if there are numbers A and B such $0 < A <. |S' - S'_n|/|S - S_n| <. B$. The notation "$<.$" means that $<$ holds for all sufficiently large n. If "$*$" denotes any relation, "$*.$" will be used in the same manner, while "$*:$" means that $*$ holds for infinitely many positive integers n.

Various methods, found in the literature, for obtaining a series $\Sigma a'_n \in MR(\Sigma a_n)$ may be summarized as follows. A sequence $\{b_n\}$ is proposed, and then the partial sums S'_n are specified by the equation $S'_n = S_n + b_{n+1}$ for $n \geqq 0$. It is immediate that $a'_0 = a_0 + b_1$, and $a'_n = a_n + b_{n+1} - b_n$ for $n \geqq 1$.

It seems somewhat advantageous to set $b_n = a_n \alpha_n$ for $n \geqq 1$, and specify the "transform sequence" $\{\alpha_n\}$. In doing so, we set $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1}$ for $n \geqq 0$, $a_{\alpha 0} = S_{\alpha 0} = a_0 + a_1\alpha_1$, and $a_{\alpha n} = S_{\alpha n} - S_{\alpha(n-1)} = a_n + a_{n+1}\alpha_{n+1} - a_n\alpha_n$ for $n \geqq 1$. If $\Sigma a_{\alpha n}$ converges, its sum will be denoted by S_α.

Suppose that Σa_n converges and $a_n \neq 0$ for $n \geqq 0$. Then with $\alpha_{n+1} = (S - S_n)/a_{n+1}, n \geqq 0$, we have $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1} = S_n + a_{n+1}(S - S_n)/a_{n+1} = S$ for $n \geqq 0$. Hence, if $MR(\Sigma a_n)$ is nonvoid, this transform sequence is the most desirable solution to our problem of speed-up. In general we must satisfy ourselves with an approximation to this solution.

For each n such that $a_{n-1} \neq 0$ we write $r_n = a_n/a_{n-1}$ and $r = \lim r_n$. Similarly, $r'_n = a'_n/a'_{n-1}$ and $r' = \lim r'_n$.

Aitken's δ^2-process can be obtained by defining its transform sequence $\{\delta_n\}$ as follows:

$$(1.1) \qquad \delta_n = 1/(1 - r_n) \text{ if } r_n \neq 1 \text{ exists; } \delta_n = 0 \text{ otherwise.}$$

The notation in (1.1) will be adhered to throughout this paper. The transform sequence $\{\alpha_n\}$ where

$$(1.2) \qquad \alpha_n = 1/(1 - r)\,,$$

being closely related to (1.1), is also considered in § 2 of this paper and in § 3.

Among publications in which (1.1) is found are the following: Aitken [1, p. 301], Forsythe [3, p. 310], Hartree [4, p. 233], Householder [5, p. 117], Isakson [6, p. 443], Lubkin [9, p. 228], Marx [10], Pflanz [11, p. 27], Samuelson [12, p. 131], Schmidt [13, p. 376], Shanks [14, p. 3], Todd [15, pp. 5, 86, 115, 187, 197, 260], and Tucker [16]. We find (1.2) in Lubkin [9, p. 232] and Shanks [14, p. 39]. Todd [15, p. 5] states that the δ^2-process dates back at least to Kummer [8].

Aitken's δ^2-process can be formulated in various ways. In particular, assuming that division by zero is excluded, we have:

$$S_{\delta n} = S_n + a_{n+1}\delta_{n+1} = S_n + a_{n+1}/(1 - r_{n+1}),\ n \geqq 0\ .$$
$$S_{\delta n} = (S_{n-1}S_{n+1} - S_n^2)/(S_{n-1} - 2S_n + S_{n+1}),\ n \geqq 1\ .$$
$$S_{\delta n} = \begin{vmatrix} \Delta S_{n-1} & \Delta S_n \\ S_{n-1} & S_n \end{vmatrix} \div \begin{vmatrix} \Delta S_{n-1} & \Delta S_n \\ 1 & 1 \end{vmatrix},\ n \geqq 1\ .$$
$$S_{\delta n} = S_{n-1} - (\Delta S_{n-1})^2/\Delta^2 S_{n-1},\ n \geqq 1\ .$$
$$S_{\delta n} = S_n - (\Delta S_{n-1}\Delta S_n)/\Delta^2 S_{n-1},\ n \geqq 1\ .$$
$$S_{\delta n} = S_{n+1} - (\Delta S_n)^2/\Delta^2 S_{n-1},\ n \geqq 1\ .$$

Returning to the most desirable solution for speed-up $\alpha_n = (S - S_{n-1})/a_n$, $n \geqq 1$, we have $\alpha_n = (a_n + (S - S_n))/a_n = 1 + (S - S_n)/a_n = 1 + T_{n+1}$, if we set $T_{n+1} = (S - S_n)/a_n$ for $n \geqq 1$. Hence $1 + T_{n+1}$, $n \geqq 1$, is the most desirable solution.

Suppose that Σa_n converges and n is any integer $\geqq 1$ such that $a_{n-1} \neq 0$. We then formally define $T_n = (S - S_{n-1})/a_{n-1}$. Similarly, $T'_n = (S' - S'_n)/a'_n$. Some relations satisfied by the quantities T_n, assuming division by zero excluded, are:

$$T_n = r_n(1 + T_{n+1})\ .$$
$$(1 - r_n)(1 + T_{n+1}) = 1 + T_{n+1} - T_n\ .$$
$$[(1 - r_n)/a_n](S - S_{n-1}) = 1 + T_{n+1} - T_n\ .$$
$$T_{n+1} = r_n/(1 - r_n) + (T_{n+1} - T_n)/(1 - r_n)\ .$$
$$T_n = r_n + r_n r_{n+1} + \cdots + (r_n r_{n+1} \cdots r_{n+k}) + \cdots\ .$$

In treating slowly convergent series Σa_n, Bickley and Miller [2] saw fit to single out the quantities $M(n)$ which in our notation is T_{n+1}, but their considerations were directed along somewhat different lines from ours and were restricted to series with positive terms only, with the additional restriction that $a_n/a_{n-1} \to 1$.

2. Acceleration, convergence or divergence, and the δ^2-process.

All series are assumed to be complex unless explicitly stated to the contrary.

THEOREM 2.1. *The conditions* (1) $r_n \to 0$, (2) $T_n \to 0$, *and* (3) $T_n/r_n \to 1$ *are equivalent.*

Proof. If $T_n \to 0$, then $a_n \neq. 0$ so that $r_n =. T_n/(1 + T_{n+1}) \to 0$. Conversely, assume that $r_n \to 0$. Let $0 < \varepsilon < 1$. Then $|r_n| \leqq. \varepsilon$, so that $|T_n| =. |r_n + r_n r_{n+1} + \cdots| \leqq. |r_n| + |r_n||r_{n+1}| + \cdots \leqq. \varepsilon/(1-\varepsilon)$ and thus $T_n \to 0$.

If $T_n \to 0$, then $T_n/r_n =. 1 + T_{n+1} \to 1$. Conversely, if $T_n/r_n \to 1$, then $T_{n+1} =. T_n/r_n - 1 \to 0$.

THEOREM 2.2. *If $T_n \to t$ for some complex number t, then:*

(1) $r = t/(1+t)$, $|r| \leqq 1$, *and* $r \neq 1$.

(2) $t = r/(1-r)$ *and* $-1/2 \leqq \operatorname{Re} t$.

If, in addition, $\{\alpha_n\}$ is a sequence of complex numbers such that $\alpha_n \to \alpha_0$ for some complex number α_0, then:

(3) $S_\alpha = S$.

(4) $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ *if and only if* $\alpha_0 = 1/(1-r)$.

(5) $\Sigma a_{\alpha n}$ *converges with the same rapidity as* Σa_n *if and only if* $\alpha_0 \neq 1/(1-r)$.

Proof. Since $\{T_n\}$ converges and $T_n =. r_n(1+T_{n+1})$, $T_n \neq. 0$ and $T_n \neq. -1$. Consequently $t \neq -1$, since otherwise $|r_n| =. |T_n/(1+T_{n+1})| \to +\infty$, which is impossible since $a_n \to 0$. Thus, $r_n =. T_n/(1+T_{n+1}) \to t/(1+t)$, i.e., $r = t/(1+t) \neq 1$. Clearly, $|r| \leqq 1$ so that (1) holds. From (1), $t = r/(1-r)$ and $|t|/|(-1)-t| = |t/(1+t)| = |r| \leqq 1$. Thus, $|t| \leqq |(-1)-t|$, which is equivalent to $-1/2 \leqq \operatorname{Re} t$, so that (2) holds. (3) holds since $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1} \to S + 0\alpha_0 = S$. Since $T_n \neq. 0$, we have $(S - S_{n-1}) \neq. 0$. If $t = 0$, then $r_n/T_n \to 1 = 1 - r$, according to (1), (2) and Theorem 2.1. If $t \neq 0$, then $r_n/T_n \to r/t = (1-r)$ from (1) and (2). In either case,

$$\begin{aligned}(S - S_{\alpha n})/(S - S_n) &=. [S - (S_n + a_{n+1}\alpha_{n+1})]/(S - S_n)\\ &=. 1 - a_{n+1}\alpha_{n+1}/(S - S_n) =. 1 - \alpha_{n+1}r_{n+1}/T_{n+1} \to 1 - \alpha_0(1-r)\ .\end{aligned}$$

Hence, (4) and (5) hold, since $1 - \alpha_0(1-r) = 0$ is equivalent to $\alpha_0 = 1/(1-r)$.

COROLLARY 2.3. *If $\{T_n\}$ converges, then $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.*

Proof. Suppose $T_n \to t$. From (1) of Theorem 2.2, $r_n \to r$ where $r \neq 1$. Thus $\delta_n =. 1/(1-r_n) \to 1/(1-r)$, so that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ according to (4) of Theorem 2.2.

We inquire if the convergence of $\{T_n\}$ is also necessary for $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. In Tucker [17], it is proven that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $T_{n+1} - T_n \to 0$.

THEOREM 2.4. *If Σa_n and $\Sigma a_{\delta n}$ are convergent real series, then $S = S_\delta$.*

Proof. Assume that $S \neq S_\delta$. Since $a_n\delta_n =. S_{\delta(n-1)} - S_{(n-1)} \to S_\delta - S \neq 0$, $\delta_n \neq. 0$ and $a_n/(1-r_n) =. a_n\delta_n \to S_\delta - S \neq 0$. Thus $a_n \to 0$ implies that $1 - r_n \to 0$, i.e., $r_n \to r = 1$ so that $0 <. r_n$ and $0 <. T_n$. From $1 + T_{n+1} - T_n =. [(1-r_n)/a_n](S - S_{n-1}) \to 0$, we have $1 + T_{n+1} - T_n <. 1/2$ and $0 <. T_{n+1} <. T_n$, which implies that $\{T_n\}$ converges.

From (1) of Theorem 2.2, $r \neq 1$, which contradicts $r = 1$. Thus our assumption is false, and $S = S_\delta$.

Lubkin [9, Th. 1] gave the first published proof of Theorem 2.4 for real series. The proof of this theorem for the complex case is given in Theorem 2.6, after the following preliminary theorem is first proved.

THEOREM 2.5. *If* $(1 - r_n)/a_n \to L \neq 0$, *then* Σa_n *diverges.*

Proof. Assume that Σa_n converges. We may suppose that $L = 1 - i$; since otherwise $\Sigma a'_n$ converges where $a'_n = a_n L/(1-i)$ and $(1 - r'_n)/a'_n =. (1 - r_n)/[a_n L/(1-i)] \to 1 - i$. Accordingly, $(1 - r_n)/a_n =. [(\mathrm{Re}\, a_n)/|a_n|^2 - (\mathrm{Re}\, a_{n-1})/|a_{n-1}|^2] + i[(\mathrm{Im}\, a_{n-1})/|a_{n-1}|^2 - (\mathrm{Im}\, a_n)/|a_n|^2] \to 1 - i$. Consequently, $(\mathrm{Re}\, a_{n-1})/|a_{n-1}|^2 <. (\mathrm{Re}\, a_n)/|a_n|^2$ so that $(\mathrm{Re}\, a_n)/|a_n|^2 \to L_1$ for some $L_1 \leqq +\infty$. If $L_1 < +\infty$, then $\mathrm{Re}\,[(1 - r_n)/a_n] \to L_1 - L_1 = 0$, which is impossible since $\mathrm{Re}\,[(1 - r_n)/a_n] \to 1$. Thus $L_1 = +\infty$ and $0 <. \mathrm{Re}\, a_n$. Similarly, $(\mathrm{Im}\, a_{n-1})/|a_{n-1}|^2 <. (\mathrm{Im}\, a_n)/|a_n|^2$ and $0 <. \mathrm{Im}\, a_n$. Hence setting $a_n = |a_n| e^{i\theta_n}$ we may chose θ_n such that $0 <. \theta_n <. \pi/2$. From

$$\begin{aligned} T_n &=. a_n/a_{n-1} + a_{n+1}/a_{n-1} + \cdots + a_{n+k}/a_{n-1} + \cdots \\ &=. |a_n/a_{n-1}| e^{i(\theta_n - \theta_{n-1})} + |a_{n+1}/a_{n-1}| e^{i(\theta_{n+1} - \theta_{n-1})} + \cdots \\ &=. [|a_n| \cos(\theta_n - \theta_{n-1}) + \cdots + |a_{n+k}| \cos(\theta_{n+k} - \theta_{n-1}) \\ &\qquad + \cdots]/|a_{n-1}| + (\mathrm{Im}\, T_n)i \end{aligned}$$

and $0 <. \theta_n <. \pi/2$, we have $0 <. \mathrm{Re}\, T_n$. Since $1 + T_{n+1} - T_n =. [(1 - r_n)/a_n](S - S_{n-1}) \to 0$, we have $1 + \mathrm{Re}\, T_{n+1} - \mathrm{Re}\, T_n =. \mathrm{Re}\,(1 + T_{n+1} - T_n) \to 0$. Thus $\mathrm{Re}\, T_{n+1} - \mathrm{Re}\, T_n < -1/2$ for $n \geqq N$, where N is some positive integer. It follows that

$$\mathrm{Re}\, T_{N+n} =. \mathrm{Re}\, T_N + \sum_{i=1}^{n} \mathrm{Re}\,[T_{N+i} - T_{N+i-1}] <. \mathrm{Re}\, T_N - \frac{n}{2} \to -\infty$$

as $n \to \infty$. Hence, $\mathrm{Re}\, T_n <. 0$ which contradicts $0 <. \mathrm{Re}\, T_n$. Consequently our initial assumption cannot hold, i.e., Σa_n must diverge.

THEOREM 2.6. *If* Σa_n *and* $\Sigma a_{\delta n}$ *both converge, then* $S = S_\delta$.

Proof. Assume that $S \neq S_\delta$. Then $a_n \delta_n =. S_{\delta(n-1)} - S_{n-1} \to S_\delta - S \neq 0$ so that $\delta_n \neq. 0$ and $a_n/(1 - r_n) =. a_n \delta_n \to S_\delta - S \neq 0$. Thus $(1 - r_n)/a_n \to 1/(S_\delta - S) \neq 0$, which implies, in view of Theorem 2.5, that Σa_n diverges, a contradiction. Therefore our assumption cannot hold, i.e., $S = S_\delta$.

After establishing the following lemma, we turn to a generalization of Theorem 2.5, using a different approach in its proof.

LEMMA 2.7. *Suppose that Σa_n is a convergent series, $a_n \neq. 0$, and $c_n = c + S_n - S$ for $n \geqq 0$ where c is some complex number. Then,*

$$1 + c\left(\frac{1 - r_n}{a_n}\right) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n} =. \frac{1 - r_n}{a_n}(S - S_{n-1}) .$$

Proof. We have

$$1 + c\left(\frac{1 - r_n}{a_n}\right) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n} =. 1 + c\left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right) + \frac{c + S_{n-1} - S}{a_{n-1}}$$
$$- \frac{c + S_n - S}{a_n} =. 1 + \frac{S - S_n}{a_n} - \frac{S - S_{n-1}}{a_{n-1}} =. \frac{S - S_{n-1}}{a_n} - \frac{S - S_{n-1}}{a_{n-1}}$$
$$=. \left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right)(S - S_{n-1}) =. \left(\frac{1 - r_n}{a_n}\right)(S - S_{n-1}) .$$

THEOREM 2.8. *If $\{(1 - r_n)/a_n\}$ is bounded, then the complex series Σa_n diverges.*

Proof. Assume that Σa_n converges. Since $\{(1 - r_n)/a_n\}$ is bounded, there is an $\varepsilon > 0$ such that $|\varepsilon(1 - r_n)/a_n| <. 1/4$. Let c be any complex number satisfying $|c| = \varepsilon$ so that

$$(1) \qquad -\mathrm{Re}\, c(1 - r_n)/a_n <. 1/4 .$$

Setting $c_n = c + S_n - S$, for $n \geqq 0$, we have $c_n \to c$. From Lemma 2.7,

$$\mathrm{Re}\left[1 + c\left(\frac{1 - r_n}{a_n}\right) + \frac{c_{n-1}}{a_{n-1}} - \frac{c_n}{a_n}\right] =. \mathrm{Re}\,\frac{1 - r_n}{a_n}(S - S_{n-1}) \to 0$$

and thus,

$$(2) \qquad 1 + \mathrm{Re}\, c\left(\frac{1 - r_n}{a_n}\right) + \mathrm{Re}\,\frac{c_{n-1}}{a_{n-1}} - \mathrm{Re}\,\frac{c_n}{a_n} <. 1/4 .$$

Using (1) and (2),

$$1/2 + \mathrm{Re}\,\frac{c_{n-1}}{a_{n-1}} <. \mathrm{Re}\,\frac{c_n}{a_n} - \mathrm{Re}\, c\left(\frac{1 - r_n}{a_n}\right) - 1/4 <. \mathrm{Re}\,\frac{c_n}{a_n} ,$$

from which it is easily seen that $\mathrm{Re}\, c_n/a_n \to +\infty$ and $\mathrm{Re}\, c_n/a_n >. 0$. Since $\mathrm{Re}\, c_n/a_n >. 0$ and $c_n \to c$, we conclude that

$$(3) \qquad a_n \notin . \{z\colon \arg c + 3\pi/4 \leqq \arg z \leqq \arg c + 5\pi/4\} .$$

Choosing $\arg c$ successively in (3) as $0, \pi/2, \pi$, and $3\pi/2$, we conclude that a_n is not in the complex plane for large n, which is absurd. Hence, our initial assumption cannot hold, i.e., Σa_n must diverge.

A proof of Theorem 2.8 can be found in the proof of a lemma by Marx [10], under the additional hypothesis that a_n is real and $a_{n-1} > a_n > 0$ for all n. His lemma is shown to contain a minor error in Tucker [16] where appropriate changes are indicated and similar comments are made on a paper by King [7].

For the series Σa_n where $a_n = 1/(\log n)$ for $n \geqq 2$, we have $(1 - r_n)/a_n \to 0$ so that, from Theorem 2.8, Σa_n diverges. Similarly, with $a_n = 1/(n + 1)$ for $n \geqq 0$, we have $1/a_n - 1/a_{n-1} = (n+1) - n = 1$ for $n \geqq 1$, and thus Σa_n diverges. For the divergent series Σa_n where $a_n = 1/(n \log n)$ for $n \geqq 2$, we have $1/a_n - 1/a_{n-1} \to \infty$, so that Theorem 2.8 is not applicable. As a final application, Theorem 2.8 manifests the divergence of the series Σa_n where $a_n = e^{i\phi_n}/(n + 1)$, $\phi_n = 1 + 1/2 + \cdots + 1/(n + 1)$, since it is easily seen that $\{1/a_n - 1/a_{n-1}\}$ is bounded.

The following theorem furnishes a generalization of Theorem 1(i), given in Tucker [16].

THEOREM 2.9. *If Σa_n is a convergent series, then some subsequence of $\{S_{\delta n}\}$ converges to S.*

Proof. Suppose Σa_n is convergent and assume that no subsequence of $\{S_{\delta n}\}$ converges to S. Since $S_{\delta n} - S_n = a_{n+1}\delta_{n+1}$, our assumption holds if and only if no subsequence of $\{a_n\delta_n\}$ converges to zero, and this is equivalent to $|a_n\delta_n| >. B$ for some $B > 0$. Thus $|(1 - r_n)/a_n| =. 1/|a_n\delta_n| <. 1/B$. From Theorem 2.8, Σa_n diverges, a contradiction. Therefore our assumption cannot be true, i.e., some subsequence of $\{S_{\delta n}\}$ converges to S.

Theorem 2.9 clearly yields a second proof of Theorem 2.6.

EXAMPLE 2.10. It is not necessarily true that if Σa_n converges, $\Sigma a_{\delta n}$ will also converge. In particular, Lubkin [9, p. 240] considers the series $\Sigma a_n = 1 + 1/2 - 1/3 - 1/4 + 1/5 + 1/6 - 1/7 - 1/8 + 1/9 + \cdots$ which converges while $\Sigma a_{\delta n}$ diverges. However, according to Theorem 2.9 some subsequence of $\{S_{\delta n}\}$ must converge to S. Here, of course, this is evident since $r_n <: 0$ and $S_{\delta n} =. S_n + a_{n+1}/(1 - r_{n+1})$. This particular series shows that the δ^2-process is not regular.

EXAMPLE 2.11. Lubkin [9, p. 240] also shows that the series $\Sigma a_n = 1 + 1/(1 + 1) + 1/2^2 + 2^2/(2^4 + 1) + 1/3^2 + 3^2/(3^4 + 1) + \cdots$ converges while $\Sigma a_{\delta n}$ diverges. Again, according to Theorem 2.9, some subsequence of $\{S_{\delta n}\}$ must converge to S. This is not so obvious by inspection as was the case in Example 2.10.

THEOREM 2.12. *If Σa_n is a series such that $\Sigma a_{\delta n}$ is properly divergent, i.e., $|S_{\delta n}| \to \infty$, as $n \to \infty$, then Σa_n diverges.*

Proof. Assume that Σa_n is convergent. From Theorem 2.9 some subsequence of $\{S_{\delta n}\}$ converges to S, so that $|S_{\delta n}| \not\to \infty$ as $n \to \infty$, i.e., $\Sigma a_{\delta n}$ is not properly divergent.

3. Acceleration and rapidity of convergence.

THEOREM 3.1. *A necessary and sufficient condition that $\{T_n\}$ converge is that $r_n \to r \neq 1$ and $T_{n+1} - T_n \to 0$.*

Proof. The necessity follows from (1) of Theorem 2.2 and the fact that $\{T_n\}$ converges implies that $T_{n+1} - T_n \to 0$.

For the sufficiency, $r \neq 1$ implies that $r_n(1 - r_n) \neq . 0$. Consequently, $T_{n+1} = . r_n/(1 - r_n) + (T_{n+1} - T_n)/(1 - r_n) \to r/(1 - r)$.

THEOREM 3.2. *If $r_n \to r$ where $|r| < 1$, then $T_n \to r/(1 - r)$.*

Proof. Since $|r| < 1, r \neq 1$ and Σa_n converges, so that T_n exists for large n. Let $\varepsilon > 0$ and ρ be any number such that $|r| < \rho < 1$. There exists an integer N such that for $n \geqq N$ and $m \geqq N$ we have $|r_n| < \rho$ and $|r_m - r_n| < \varepsilon(1 - \rho)$. Thus, for each $n \geqq N$ we have

$$\begin{aligned}|T_{n+1} - T_n| &= |[r_{n+1} - r_n] + [r_{n+1}r_{n+2} - r_n r_{n+1}] + \cdots \\ &\quad + [(r_{n+1} \cdots r_{n+k+1}) - (r_n \cdots r_{n+k})] + \cdots | \\ &\leqq |r_{n+1} - r_n| + |r_{n+1}||r_{n+2} - r_n| + \cdots \\ &\quad + |r_{n+1} \cdots r_{n+k}||r_{n+k+1} - r_n| + \cdots \\ &< \varepsilon(1 - \rho) + \rho\varepsilon(1 - \rho) + \cdots + \rho^k\varepsilon(1 - \rho) + \cdots = \varepsilon .\end{aligned}$$

Hence, $|T_{n+1} - T_n| \to 0$, i.e., $T_{n+1} - T_n \to 0$. From Theorem 3.1, $\{T_n\}$ converges. Consequently, $T_n \to r/(1-r)$ according to (2) of Theorem 2.2.

THEOREM 3.3. *Suppose that $r_n \to r$ where $|r| < 1$, and let $\{\alpha_n\}$ be a complex sequence converging to some complex number α_0. Then $T_n \to t$ for some complex number t, and conditions* (1) *through* (5) *of Theorem* 2.2 *hold.*

Proof. From Theorem 3.2, $\{T_n\}$ converges. We now apply Theorem 2.2.

According to Theorem 3.3, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if $r = 0$. Nevertheless, the reader should be forewarned in case $r = 0$. In particular, let $\Sigma a_n = \sum_0^\infty (-1)^n/n! = 1/e$. We have $r_n = -1/n$ for $n \geqq 1$, and $\delta_n = 1/(1 - r_n) = 1/[1 + (1/n)] = n/(n+1) = 1 - 1/(n+1) = 1 + r_{n+1}$ for $n \geqq 2$. Consequently, $S_{\delta n} = S_n + a_{n+1}\delta_{n+1} = S_n + a_{n+1}(1 + r_{n+2}) = S_{n+2}$ for $n \geqq 1$.

LEMMA 3.4. *If* $|r| < 1$, *then* $T_n/r_n \to 1/(1 - r)$.

Proof. If $r = 0$, then $T_n/r_n \to 1 = 1/(1 - r)$ according to Theorem 2.1. If $r \neq 0$, then $T_n/r_n \to [r/(1 - r)]/r = 1/(1 - r)$ according to Theorem 3.2.

THEOREM 3.5. *Suppose that* Σa_n *and* $\Sigma a'_n$ *are series such that* $|r| < 1$ *and* $|r'| < 1$. *Then*:

(1) $\Sigma a'_n$ *converges more rapidly than* Σa_n *if and only if* $a'_n/a_n \to 0$.

(2) $\Sigma a'_n$ *converges with the same rapidity as* Σa_n *if and only if there are numbers* a *and* b *such that* $0 < a <. |a'_n/a_n| <. b$.

Proof. From Lemma 3.4, $T_n/r_n \to 1/(1 - r)$ and $T'_n/r'_n \to 1/(1 - r')$. If $a'_n/a_n \to 0$,

$$\frac{S' - S'_{n-1}}{S - S_{n-1}} =. \frac{a'_n}{a_n}\frac{T'_n/r'_n}{T_n/r_n} \to 0 \cdot \frac{1/(1 - r')}{1/(1 - r)} = 0 .$$

Conversely, if $\Sigma a'_n$ converges more rapidly than Σa_n,

$$\frac{a'_n}{a_n} =. \frac{T_n/r_n}{T'_n/r'_n}\frac{S' - S'_{n-1}}{S - S_{n-1}} \to \frac{1/(1 - r)}{1/(1 - r')} \cdot 0 = 0 .$$

This proves (1).

Assume that a and b are numbers such that $0 < a <. |a'_n/a_n| <. b$. Since $|(T'_n/r'_n)/(T_n/r_n)| \to |(1 - r)/(1 - r')| \neq 0$, there are numbers c and d such that $0 < c <. |(T'_n/r'_n)/(T_n/r_n)| <. d$. Thus,

$$0 < ac <. \left|\frac{S' - S_{n-1}}{S - S_{n-1}}\right| =. \left|\frac{a'_n}{a_n}\right| \left|\frac{T'_n/r'_n}{T_n/r_n}\right| <. bd .$$

Assume that A and B are numbers such that

$$0 < A <. |(S' - S'_{n-1})/(S - S_{n-1})| <. B$$

As above, there are numbers c and d such that

$$0 < c <. |(T_n/r_n)/(T'_n/r_n)| <. d .$$

Thus,

$$0 < Ac <. \left|\frac{a'_n}{a_n}\right| =. \left|\frac{T_n/r_n}{T'_n/r'_n}\right| \left|\frac{S' - S'_{n-1}}{S - S_{n-1}}\right| <. Bd.$$

LEMMA 3.6. *If* $|r_n| \leqq . \rho < 1/2$ *for some number* ρ, *then*

$$0 < (1-2\rho)/(1-\rho) \leqq . |T_n/r_n| \leqq . 1/(1-\rho) .$$

Proof. We have $|T_n| \leqq . |r_n| + |r_n r_{n+1}| + \cdots + |r_n \cdots r_{n+k}| + \cdots \leqq . |r_n|/(1-\rho) \leqq . \rho/(1-\rho) < 1$. Thus, $|T_n/r_n| \leqq . 1/(1-\rho)$ and $|T_n/r_n| = . |1 + T_{n+1}| \geqq . ||1| - |T_{n+1}|| = . 1 - |T_{n+1}| \geqq . 1 - \rho/(1-\rho) = (1-2\rho)/(1-\rho) > 0$.

THEOREM 3.7. *Suppose that* Σa_n, $\Sigma a'_n$ *are series such that* $a'_n/a_n \to 0$, *and* $|r_n| \leqq . \rho_1 < 1/2$, $|r'_n| \leqq . \rho_2 < 1$ *for some numbers* ρ_1, ρ_2. *Then* $\Sigma a'_n$ *converges more rapidly than* Σa_n.

Proof. From Lemma 3.6, $0 < (1-2\rho_1)/(1-\rho_1) \leqq . |T_n/r_n|$. Also, $|T'_n/r'_n| = . |1 + r'_{n+1} + r'_{n+1}r'_{n+2} + \cdots| \leqq . 1/(1-\rho_2)$. Thus,

$$\frac{|S' - S'_{n-1}|}{|S - S_{n-1}|} = . \frac{|a'_n|}{|a_n|} \frac{|T'_n/r'_n|}{|T_n/r_n|} \leqq . \frac{|a'_n|}{|a_n|} \frac{1/(1-\rho_2)}{(1-2\rho_1)/(1-\rho_1)} \to 0 .$$

The following counterexample shows that the hypothesis of Theorem 3.7 cannot be relaxed by replacing 1/2 by any larger number.

COUNTEREXAMPLE 3.8. Let ε be any number such that $0 < \varepsilon < 1/4$ and $f(x, n) = x^{n+1} - 2x + 1$, $n = 1, 2, \cdots$. Then $f(1/2, n) >. 0$ and $f(1/2 + \varepsilon, n) <. 0$. We may thus assume that N is a positive integer such that for some b, $f(b, N) = 0$ and $1/2 < b < 1/2 + \varepsilon$. Thus, $-1 + b + b^2 + \cdots + b^N = (b-1)^{-1} f(b, N) = 0$. Define $a_n = -b^n$ for $n = k(N+1)$ and $k = 0, 1, 2, \cdots$, and $a_n = b^n$ otherwise. Accordingly, Σa_n converges, $|r_n| = b < 1/2 + \varepsilon$ and $S - S_n =: 0$. Hence the series $\Sigma a'_n$, where $a'_n = a_n/n!$, $a'_n/a_n \to 0$ and $|r'_n| \to 0$, does not converge more rapidly than Σa_n.

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