MULTIPLIERS AND $H^*$ ALGEBRAS

WAI-MEE CHING AND JAMES SAI-WING WONG
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Let $A$ be a normed algebra and $B(A)$ the algebra of all bounded linear operators from $A$ into itself, with operator norm. An element $T \in B(A)$ is called a multiplier of $A$ if $(Tx)y = x(Ty)$ for all $x, y \in A$. The set of all multipliers of $A$ is denoted by $M(A)$. In the present paper, it is first shown that $M(A)$ is a maximal commutative subalgebra of $B(A)$ if and only if $A$ is commutative. Next, $M(A)$ in case $A$ is an $H^*$-algebra will be represented as the algebra of all complex-valued functions on certain discrete space. Finally, as an application of the representation theorem of $M(A)$, the set of all compact multipliers of compact $H^*$-algebras is characterized.

In case $A$ is commutative, the general notion of multipliers was first studied by Helgason [7], followed by Wang [12] and Birtel [2], [3], [4]. In the special case when $A = L_1(G)$, the group algebra over an arbitrary locally compact abelian group, the problem of multipliers has also been studied by Helson [8] and Edwards [5]. (Cf. also Rudin [11].) Helgason [7] called a function $g$ on the maximal ideal space $\hat{A}$ of $A$ a multiplier if $gA \subseteq \hat{A}$ where $\hat{A}$ is the Gelfand transform of $A$. Later Wang [12] and Birtel [2] carried out more systematic studies on multipliers. In case $A$ is semi-simple, Wang [12] proved that there exists a norm-decreasing isomorphism between $M(A)$ and $C_0(\hat{A})$, the algebra of bounded continuous functions of $\hat{A}$. In particular if $A = L_1(G)$, then $M(A) = M(G)$, the algebra of all bounded regular Borel measures on $G$. In the noncommutative case, Wendel [13] first studied multipliers for noncommutative group algebras, followed by Kellogg [9] for $H^*$-algebras. However, since Kellogg’s proofs rely heavily on the representation theorem of Wang [12] for multipliers on general commutative semi-simple Banach algebras, relevant results on multipliers of $H^*$-algebras were obtained only for the commutative case.

2. Multiplier algebras. Let $A$ be a normed algebra. $A$ is said to without order if either $xA = \{0\}$ or $Ax = \{0\}$ implies $x = 0$. Clearly, if $A$ is semi-simple or $A$ has a unit, then $A$ is without order. In the sequel, we assume all normed algebras under consideration are without order. An element $T \in B(A)$ is called a right (left) multiplier

\[1\] Both Kellogg [9] and Wendel [13] used the terminology “centralizers” instead of “multipliers".
of $A$ if $T(xy) = (Tx)y(T(xy) = x(Ty))$. We denote the set of all right (left) multipliers of $A$ by $R(A)(L(A))$. We first observe the following:

**Proposition 1.** $R(A) \cap L(A) = M(A)$.

**Proof.** Clearly, we have $R(A) \cap L(A) \subseteq M(A)$. Let $T \in M(A)$. Note that $(T(xy))z = (xy)Tz = x(y(Tz)) = x((Ty)z)$ for all $x, y, z \in A$. Since $A$ is without order, $T(xy) = x(Ty)$, i.e. $T \in R(A)$. Similarly, one easily shows that $T \in L(A)$, completing the proof.

A commutative subalgebra $Y$ of an algebra $X$ is called *maximal commutative subalgebra* of $X$ if $Y$ is not properly contained in any proper commutative subalgebra of $X$. If $X$ has an identity element $e$, $e$ belongs to any maximal commutative subalgebra of $X$. Using an argument based upon Zorn's lemma, one easily shows that $M(A)$ is contained in some maximal commutative subalgebra of $B(A)$, say $MC(A)$.

For an arbitrary normed algebra $X$, we denote its centre by $Z(X)$. One can easily verify the following inclusions:

$$Z(B(A)) \subseteq Z(M(A)) \subseteq M(A) \subseteq MC(A) \subseteq B(A).$$

Kellogg [9] proved that $M(A)$ is a closed commutative subalgebra of $B(A)$, consequently we always have $M(A) = Z(M(A))$. More precisely, we can prove the following:

**Proposition 2.** Let $A$ be a normed algebra. Then the algebra $M(A)$ of all multipliers of $A$ is a closed commutative subalgebra of $B(A)$, the algebra of all bounded linear operators in $A$ with operator norm.

**Proof.** Let $T_n \in M(A)$ and $\| T_n - T \| \to 0$, for $n = 1, 2, 3, \ldots$. We note that for any $x, y \in A$,

$$\| x(Ty) - (Tx)y \| \leq \| x(Ty) - x(T_n y) \| + \| (T_n x)y - (Tx)y \| \leq 2 \| x \| \| y \| \| T_n - T \|. $$

Letting $n$ tend to infinity, we have $x(Ty) = (Tx)y$. Thus $T \in M(A)$, and $M(A)$ is closed. These remarks together with the result of Kellogg complete the proof of the assertion.

From Proposition 2, we may easily deduce that all subalgebras of $B(A)$ occurring (*) are closed in $B(A)$.

**Proposition 3.** Let $S^A_x$ denote the spectrum of an element $x \in A$. Then $S^B(A)(T) = S^M(A)(T)$.
Proof. Since both $B(A)$ and $M(A)$ contain the identity, we need only to prove that for $T \in M(A)$ if $T^{-1}$ exists and is in $B(A)$, then $T^{-1} \in M(A)$. For any $x, y \in A$, we observe that

$$(T^{-1}x)y = (T^{-1}x)(TT^{-1}y) = (TT^{-1}x)(T^{-1}y) = x(T^{-1}y) ,$$

Theorem 1. $M(A)$ is maximal commutative subalgebra of $B(A)$ if and only if $A$ is commutative.

Proof. Let $A$ be commutative, and for each $x \in A$, we write $T_x$ the left and right regular representations of $x$ in $B(A)$. Since $A$ is commutative, $[A] = \{T_x: x \in A\} = \{xT: x \in A\} \subseteq M(A)$. Suppose $A$ is not maximal, and let $MC(A)$ be some maximal commutative subalgebra containing $A$. Since $A$ is not maximal, we may pick $T \in MC(A) \setminus M(A)$. On the other hand, $T \in MC(A)$ implies that $T$ commutes with all elements of $[A]$, i.e., for all $x, y \in A$ $(Tx)y = (TT_y)y = (T_yT)y = x(Ty)$, proving that $T \in M(A)$. This contradiction establishes that $A$ is maximal. Conversely let $M(A)$ be a maximal commutative algebra. Thus $T \in B(A)$, and $ST = TS$ for all $S \in M(A)$ imply $T \in M(A)$. In particular, $(T_x S)y = x(Sy) = (SX)y = (ST_x)y$ and hence $T_x \in M(A)$ for all $x \in A$. Thus $(xy)z = T_x(yz) = y(T_xz) = (yx)z$ for all $x, y, z \in A$. Since $A$ is without order, $xy = yx$ for all $x, y \in A$, i.e., $A$ is commutative.

We will see from §3 and §4 that in case $A$ is a simple $H^*$-algebra, then $M(A) = Z(B(A))$.

Remark 1. If $A$ is in addition complete, then $M(A)$ is also a Banach algebra. In this case, we may define $T \in M(A)$ as any mapping of $A$ into itself satisfying the condition that $(Tx)y = x(Ty)$ for all $x, y \in A$. From the fact that $A$ is without order, it is easily seen that $T$ is linear. As a consequence of closed graph theorem, we may also show that $T$ is bounded (see Wang [12]). The way we choose to define multipliers is just a matter of convenience. Note that throughout all of our discussion, we do not assume $A$ to be complete.

3. Lemmata on matrix algebras. Let $X_s$ be the algebra of all matrices $(x_{a\beta})$, $\alpha, \beta \in S$, where $S$ is a fixed set of indices and $x_{a\beta}$'s are complex numbers satisfying the condition $\sum_{a, \beta} |x_{a\beta}|^2 < \infty$. The multiplication is defined by

$$z = (z_{a\beta}) = x \cdot y = (x_{a\beta})(y_{\gamma\delta}) ,$$

where

$$z_{a\beta} = \sum_{\gamma \in S} x_{a\gamma} y_{\gamma\beta} .$$
This multiplication is well defined since
\[ \sum_{\alpha, \beta} |Z_{\alpha\beta}|^2 = \sum_{\alpha, \beta} \left| \sum_{\ell} x_{\alpha\ell} y_{\gamma\ell} \right|^2 \leq \left( \sum_{\alpha, \beta} |x_{\alpha\beta}|^2 \right) \left( \sum_{\ell, \gamma} |y_{\gamma\ell}|^2 \right) < \infty. \]

We define an inner product on \( X_s \) by
\[ (x, y) = \sum_{\alpha, \beta} x_{\alpha\beta} y_{\gamma\delta}, \]
where \( w \) is a fixed constant \( \geq 1 \). \( X_s \) becomes a Banach algebra if the norm is induced by the inner product in the usual manner, i.e. \( ||x||^2 = (x, x) \).

In this cases, \( B(X_s) \) can be identified with a subalgebra of all matrices \( T = (t_{\alpha\beta}) \) over \( S \times S \) such that \( Tx = y \) is defined by
\[ y_{\alpha\beta} = \sum_{(i, \delta)} t_{\alpha\beta i\delta} y_{i\delta} \]
with \( \sum_{\alpha, \beta} |y_{\alpha\beta}|^2 < \infty \). (We refer to Naimark [10] for more detailed discussion of \( X_s \).)

**Lemma 1.** \( T \in M(X_s) \) if and only if \( T \) is a scalar multiple of the identity operator.

**Proof.** Let \( T = (t_{\alpha\beta}) \in M(X_s) \), so \( (Tx)y = T(xy) \) for all \( x, y \in X_s \).

For any fixed pair of indices \( (\sigma, \tau) \in S \times S \), let \( x_{\sigma\tau} = 1, x_{\alpha\beta} = 0 \) if \( (\alpha, \beta) \neq (\sigma, \tau) \) and \( y_{\sigma\tau} = 1, y_{\alpha\beta} = 0 \) otherwise. Denote \( z = (z_{\alpha\beta}) = (Tx)y = T(xy) \). Observe from \( z = (Tx)y \) that
\[ \sum_{\xi} \left( \sum_{(i, \delta)} t_{\alpha\xi i\delta} \right) (y_{\xi\delta}) = \sum_{\xi} t_{\alpha\xi\xi\delta} y_{\xi\delta}, \]
and hence \( z_{\sigma\tau} = t_{\alpha\sigma\tau}, z_{\alpha\beta} = -t_{\alpha\sigma\tau}, z_{\alpha\beta} = 0 \) otherwise. On the other hand, from \( z = T(xy) \) we have
\[ \sum_{(i, \delta)} t_{\alpha\beta i\delta} \left( \sum_{\xi} x_{\xi\ell} y_{\gamma\ell} \right) = \sum_{\sigma} t_{\alpha\beta\sigma\delta} y_{\gamma\delta} = -t_{\alpha\beta\sigma\tau}. \]

From these computation, we obtain that \( t_{\alpha\beta\sigma\tau} = 0 \) if \( \beta \neq \sigma \) and \( \beta \neq \tau \). In case \( \beta = \sigma \), we have \( t_{\alpha\beta\sigma\tau} = -t_{\alpha\sigma\tau} \) and so again \( z_{\alpha\beta} = 0 \). Hence we conclude that \( t_{\alpha\beta\sigma\tau} = 0 \) unless \( \beta = \tau \). Similarly, from \( x(Ty) = T(xy) \) we obtain \( t_{\alpha\beta\sigma\tau} = 0 \) unless \( \alpha = \sigma \). Since \( \sigma, \tau \) are arbitrary, we have \( t_{\alpha\beta\sigma\tau} = 0 \) only if \( (\alpha, \beta) = (\sigma, \tau) \). Next we choose \( x_{\sigma\tau} = 1, x_{\alpha\beta} = 0 \) if \( (\alpha, \beta) \neq (\sigma, \tau) \) and \( y_{\rho\nu} = 1, y_{\alpha\beta} = 0 \) if \( (\alpha, \beta) \neq (\mu, \nu) \) in the equation \( (Tx)y = x(Ty) \). It is readily seen from a similar computation that \( t_{\alpha\beta\gamma\delta} = t_{\gamma\delta\beta} \) for all \( \alpha, \beta, \gamma, \delta \in S \). Thus if \( T \in M(X_s) \), then \( T \) must be a scalar multiple of the identity operator.

**Lemma 2.** \( M(X_s) = Z(B(X_s)) \).

**Proof.** In view of the inclusion relation \((*)\), we need only to show that if \( T \in Z(B(X_s)) \), then \( T \in M(X_s) \). Let \( T = (t_{ij}), i, j \in S \times S, \)
such that for two fixed distinct indices $k, h \in S \times S$, $t_{kk} = a \neq t_{hh} = b$ and $t_{ij} = 0$ otherwise. From Lemma 1, we clearly have $T \in M(A)$. Define $T_1 \in B(A)$, $T_1 = (t''_{ij})$, by $t''_{kk} = 1$, and $t''_{ij} = 0$ otherwise. It is readily seen by a direct computation that $TT_1 \neq T_1 T$, hence $T \not\in Z(B(X_S))$, proving the assertion.

4. $H^*$-algebras. An $H^*$-algebra $A$ is a Banach $*$-algebra (a Banach algebra with involution) and a Hilbert space, where the Banach algebra norm coincides with the Hilbert space norm, with the crucial connecting property $(xy, z) = (y, x^*y)$. It is assumed that for each $x \in A$, $\|x^*\| = \|x\|$ and $x^*x \neq 0$ if $x \neq 0$. A simple example of an $H^*$-algebra is the matrix algebra $X_S$ introduced in §3. In fact, $X_S$ is a simple $H^*$-algebra, and indeed every simple $H^*$-algebra is isometric and $*$-isomorphic to some matrix algebra $X_S$. In general, Ambrose [1] proved that every $H^*$-algebra is the direct, and at the same time orthogonal, sum of its closed minimal two-sided ideals which are simple $H^*$-algebras. (Naimark [10], p. 331).

**Lemma 3.** Let $A$ be a normed algebra which is the direct sum of closed two-sided ideals $\{I_\alpha; \alpha \in \mathcal{E}\}$ in $A$. If $T \in M(A)$, then $T$ maps each $I_\alpha$ into itself.

**Proof.** Let $x \in I_\alpha$ for some fixed $\alpha \in \mathcal{E}$. Suppose that $(Tx)_\beta \neq 0$, i.e. The projection of $Tx$ into $I_\beta$, for some $\beta \neq \alpha, \beta \in \mathcal{E}$. We may choose $y \in I_\beta, y \neq 0$, such that $(Tx)y = (Tx)_\beta y = 0$. (For otherwise, if $(Tx)_\beta I_\beta = 0$, then

$$(Tx)_\beta A = (Tx)_\beta \left( \bigoplus_{\alpha \in \mathcal{E}} I_\alpha \right) = (Tx)_\beta I_\beta = 0,$$

contradicting the fact that $A$ is without order.) But on the other hand, $T(xy) = T \cdot 0 = 0$, violating the multiplier condition. Thus, $(Tx)_\beta = 0$, i.e. $T$ maps each $I_\alpha$ into itself.

Denote by $T_\alpha$ the restriction of $T$ to $I_\alpha$. It is clear that if $T \in M(A)$, then $T_\alpha \in M(I_\alpha)$ for each $\alpha \in \mathcal{E}$. Hence we may write

$$TA = T\left( \bigoplus_{\alpha \in \mathcal{E}} I_\alpha \right) = \bigoplus_{\alpha \in \mathcal{E}} TI_\alpha = \bigoplus_{\alpha \in \mathcal{E}} T_\alpha I_\alpha.$$

We note that for each $T \in M(A)$, there corresponds a unique set $\{T_\alpha\}$ where $T_\alpha \in M(I_\alpha)$.

**Theorem 2.** Let $A$ be an $H^*$-algebra, and $\{I_\alpha; \alpha \in \mathcal{E}\}$ the set of all minimal closed two-sided ideals in $A$. Denote by $E$ the topological space of the set of all minimal closed two-sided ideals in $A$ with the
discrete topology. Then there exists a \(\ast\)-isomorphism which is at the same time an isometry of \(M(A)\) onto \(C^\infty(E)\), the space of all bounded continuous complex functions on \(E\).

**Proof.** From the structure theorem of \(H^\ast\)-algebras, we know that \(A = \bigoplus \sum_a I_a\) of all its closed minimal ideals which are simple \(H^\ast\)-algebras, \(\ast\)-isomorphic and isometric to some matrix algebras \(X_{s_a}\). For each \(T \in M(A)\), let \(\{T_\alpha : \alpha \in \mathcal{E}\}\) be the corresponding set of multipliers of \(I_a\). By Lemma 1, \(T_\alpha\) must be a scalar multiple of the identity operator \(P_a\), say \(T_\alpha = t(\alpha)P_a\), for some complex number \(t(\alpha)\) depending on \(T\). Define \(\Phi : M(A) \to C(E)\), the space of all complex-valued functions on \(E\) by \(\Phi(T)(\alpha) = t(\alpha)\) for each \(\alpha \in E\). Clearly \(\Phi\) is linear, multiplicative and preserves involution, (i.e., \(*\) operations for elements in \(A\), complex conjugation for elements in \(C^\infty(E)\) and operator adjoint for elements in \(M(A)\).) To show that \(\Phi\) is isometric, we observe

\[
\|Tx\|^2 = \left\|\sum_{\alpha} T_\alpha x_\alpha\right\|^2 = \left\|\sum_{\alpha} T_\alpha x_\alpha\right\|^2 = \sum_{\alpha} \|T_\alpha x_\alpha\|^2 = \sum_{\alpha} \|t(\alpha)x_\alpha\|^2 \leq \|\Phi(T)\|^2 \|x\|^2
\]

and hence \(\|T\| \leq \|\Phi(T)\|\). Conversely, we have for some \(x_\alpha \neq 0\),

\[
\|\Phi(T)(\alpha)\| = |t(\alpha)| = \frac{\|T_\alpha x_\alpha\|}{\|x_\alpha\|} \leq \|T_\alpha\| \leq \|T\|
\]

proving \(\|\Phi(T)\| \leq \|T\|\). Thus, \(\Phi\) is indeed an isometry, and being linear, it is one-to-one. On the other hand for each \(f \in C^\infty(E) \subseteq C(E)\), let \(T_\alpha = f(\alpha)P_a\). It is readily seen that the mapping \(T\) determined by \(\{T_\alpha\}\) belongs to \(M(A)\) and satisfies \(\Phi(T) = f\). Thus, we conclude that \(\Phi\) is an isometric \(\ast\)-isomorphism from \(M(A)\) onto \(C^\infty(E)\).

We note that the present proof differs from its commutative counterpart [9] in the use of Ambrose's structure theorem [1] for \(H^\ast\)-algebras instead of Gelfand's representation for general commutative Banach Algebras.

**Remark 2.** We note that the orthogonal complement of each minimal closed two-sided ideal is a maximal closed two-sided ideal, and vice versa. Hence the space of all minimal closed two-sided ideals is homeomorphic to the space of all maximal closed two-sided ideals. Thus, in case \(A\) is commutative, the above representation theorem reduces to that of Kellogg's (Theorem (4.1), [9]).

**Remark 3.** From Lemma 2 and the above theorem, it is easily seen that if \(A\) is a \(H^\ast\)-algebra then \(M(A) = Z(B(A))\) if and only if \(A\) is simple.
Remark 4. The result of Theorem 2 remains valid for any algebra which is the direct sum of ideals \( \{ I_a \} \) such that each ideal is isomorphic and isometric to some matrix algebra. The isometry of \( M(A) \) and \( C^\omega(E) \) can be proved without using the orthogonality of the direct sum in an \( H^* \)-algebra.

Remark 5. Since \( M(A) \) is a commutative involutory algebra, it is also contained in the set of all normal operators on \( A \).

Remark 6. Since \( M(A) \) is \(*\)-isomorphic and isometric to \( C^\omega(E) \), its maximal ideal space is homeomorphic to the Stone-Cech compactification of the discrete space \( E \). (See [6], Chapter 6).

Remark 7. A Banach \(*\)-algebra \( A \) with identity \( e \) is called completely symmetric if for each \( x \in A, (e + x^*x)^{-1} \in A \). (See Naimark [10], p. 299.) It is clear that \( C^\omega(E) \) and hence \( M(A) \) is completely symmetric. In particular, the Shilov boundary of \( M(A) \) coincides with its maximal ideal space. (Cf. Naimark [10], p. 218.)

Another interesting example of \( H^* \)-algebras is the group algebra \( L^2(G) \), where \( G \) is an arbitrary compact group. In this case, all the minimal closed two-sided ideals of \( L^2(G) \) are isomorphic and isometric to finite dimensional simple \( H^* \)-algebras, or equivalently \( X_{S_\alpha} \), with \( S_\alpha \) finite for each \( \alpha \in \Sigma \) (see [1].). In the following, we will prove a result for the set of all multipliers which are at the same time compact operators in case \( A \) is a \( H^* \)-algebra whose minimal closed two-sided ideals are finite-dimensional. (Such an algebra will be called compact \( H^* \)-algebra. Clearly, every commutative \( H^* \) algebra is a compact \( H^* \)-algebra.)

Theorem 3. Let \( A \) be a \( H^* \)-algebra whose minimal closed two-sided ideals are finite dimensional, and \( M_0(A) \) the set of all compact operators in \( M(A) \). Then \( \Phi(M_0(A)) = C_0(E) \), the algebra of all continuous functions on \( E \) which vanish at infinity.

Proof. Since every \( I_\alpha \) is finite dimensional, each \( T_\alpha \in M(I_\alpha) \) is a scalar multiple of the identity operator \( P_\alpha \), and hence compact. For any finite set \( F \subseteq E \), if we define

\[
T = \sum_{\alpha \in F} T_\alpha = \sum_{\alpha \in F} c_\alpha P_\alpha,
\]

where \( c_\alpha \) are complex constants, \( T \) is the finite sum of compact operators and thus again compact. Let \( C_\kappa(E) \) be the algebra of all continuous functions on \( E \) with compact support. We have just seen
that $\Phi^{-1}(C_K(E)) \subseteq M_0(A)$. Since $C_K(E) = C_0(E)$, thus $\Phi^{-1}(C_K(E)) = \Phi^{-1}(C_K(E))$. However, $M_0(A)$ is the intersection of the closed sub-algebra $M(A)$ and the closed ideal of all compact operators in $B(A)$, and is thus closed. As a consequence, we have $\Phi^{-1}(C_K(E)) \subseteq M_0(A)$.

On the other hand, suppose that there exists a $T \in M_0(A)$ such that $\Phi(T) = f \in C_0(E)$, i.e., there exists $\varepsilon > 0$ such that the set $G = \{\alpha \in E : |f(\alpha)| \geq \varepsilon\}$ is infinite. For each $\alpha \in G$, choose $x_\alpha \in I_\alpha$ with $\|x_\alpha\| = 1$. Note that $\{x_\alpha\}$ is a bounded sequence, but $\{Tx_\alpha\} = \{f(\alpha)x_\alpha\}$ is an orthogonal sequence with $\|Tx_\alpha\| \geq \varepsilon$ which cannot have any convergent subsequence. This contradicts the fact that $T$ is compact. Thus, $M_0(A) \subseteq \Phi^{-1}(C_0(E))$, completing the proof.

**Remark 8.** We note that for every compact multiplier $T$ of a compact $H^*$-algebra, there exists a net $T_n \in B(A)$ with finite ranks, such that $T_n$ converges to $T$ in operator norm.

**Remark 9.** For each $T \in M(A)$, let $\{T_n\}$ be the collection of all restrictions of $T$ to $I_\alpha$. Clearly $\{T_n\}$ is a family of mutually orthogonal projections, since $\{I_\alpha\}$ is an orthogonal family of subspaces. For each $T \in M_0(A)$, we observe that there are only countably many $T_n$ different from zero. (Observe that the set $\{\alpha : f(\alpha) \neq 0, f = \Phi(T)\} = \bigcup_{n=1}^{\infty} S_n$, where $S_n = \{\alpha : |f(\alpha)| \geq 1/n\}$, is countable since for each $n$, $S_n$ is finite.) Hence, we may write

$$T = \sum_{i=1}^{\infty} f(\alpha_i)P_{I_\alpha}, \quad \text{with} \quad \lim_{i \to \infty} |f(\alpha_i)| = 0.$$ 

This decomposition of $T$ into a sequence of orthogonal projections can be considered as an extension of the well-known spectral decomposition of a self-adjoint compact operators of $H^*$-algebras. In this case, $T$ is not assumed to be self-adjoint.

**Remark 10.** By a similar consideration as given in Remark 2, Theorem 3 may be considered as a generalization of Theorem (4.3) of [9]. Furthermore, the maximal ideal space of the algebra $M_0(A)$ of all compact multipliers of a compact $H^*$-algebra $A$ is homeomorphic to $E$, the set of all minimal two-sided ideals in $A$ with discrete topology.

**Remark 11.** We remark that the specialization of general $H^*$-algebras to compact $H^*$-algebras is necessary since in case of $X_s$, the identity operator in $B(X_s)$ is compact if and only if $X_s$ is finite-dimensional.
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