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A PHRAGMÉN-LINDELÖF THEOREM FOR FUNCTION ALGEBRAS

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Let A be a function algebra, considered as a closed subalgebra of $C(\mathfrak{M})$, where \mathfrak{M} is the space of multiplicative linear functionals on A . Let ∂ denote the Šilov boundary of A . We shall call $\mathfrak{M} \setminus \partial$ the "interior of \mathfrak{M} " and say a function g on this "interior" is A -holomorphic if each φ in $\mathfrak{M} \setminus \partial$ has a neighborhood on which g is uniformly approximable by elements of A .

What we shall observe here is that results of the Phragmén-Lindelöf type apply to certain A -holomorphic functions.

These results follow easily from the type of argument used in an earlier paper [1] in which function-algebra analogues of some classical results of function theory were obtained; the present note is essentially an addendum to [1] (where " A -holomorphic" [3] was "locally approximable"). Other results of the Phragmén-Lindelöf type have been obtained by Quigley [2].

Our analogue of the usual Phragmén-Lindelöf result replaces the point at infinity by a peak set lying in the Šilov boundary.

THEOREM 1. *Suppose $f \in A$ peaks on $F \subset \partial$, and g is an A -holomorphic function defined and continuous on $\mathfrak{M} \setminus F$. Suppose g is bounded on $\partial \setminus F$ and for some α , $0 < \alpha < 1$, and $k > 0$*

$$(1) \quad g \exp\left(\frac{-k}{|1 - f|^\alpha}\right)$$

is bounded on the interior of \mathfrak{M} . Then g is bounded on $\mathfrak{M} \setminus F$ by its bound on $\partial \setminus F$.

Thus an unbounded A -holomorphic function continuous on $\mathfrak{M} \setminus F$ cannot increase too slowly as we approach F . Actually g need only be defined on $\mathfrak{M} \setminus \partial$ (and A -holomorphic) if we replace $\partial \setminus F$ by a deleted neighborhood of it in \mathfrak{M} .

THEOREM 2. *With f , F and α as above, let g be an A -holomorphic function which is bounded on the intersection V of a neighborhood of $\partial \setminus F$ with the interior of \mathfrak{M} , and suppose (1) holds. Then g is bounded by its bound on V .*

Both of these results are easy consequences of the local maximum

modulus principle [4] and classical arguments. A little more is needed for the following extension of the Phragmén-Lindelöf corollary concerning a bounded analytic function on a sector having a limit as $z \rightarrow \infty$ along the bounding rays.

THEOREM 3. *Suppose g is a bounded function on \mathfrak{M} which is A -holomorphic, has its restriction to ∂ continuous, and in fact is continuous at each point of $\partial \setminus \bigcup_{n=1}^{\infty} K_n$, where K_n is a zero set of A lying in the⁽¹⁾ Choquet boundary. Then g is continuous on \mathfrak{M} .*

Thus we cannot have too small a set of discontinuities for an A -holomorphic function which has a continuous restriction to the Šilov boundary and also is continuous at a fairly large set of points in ∂ .

As a mixture of Theorems 1 and 3 we obtain

COROLLARY 4. *Suppose g is a (not necessarily bounded) function on \mathfrak{M} which is A -holomorphic, has its restriction to ∂ continuous and is continuous at each point of $\partial \setminus \bigcup_{n=1}^{\infty} K_n$, where K_n is a zero set of A lying in the Choquet boundary. Suppose $f \in A$ peaks on K_1 while (1) is bounded on the interior of \mathfrak{M} . Then g is continuous on \mathfrak{M} .*

Proofs. Our proof of Theorem 1 is simply an imitation of a classical argument [5]. To begin let $\alpha < \beta < 1$; noting that

$$|\arg(1 - f)| \leq \pi/2,$$

we have an element $(1 - f)^\beta$ in A (where we apply the principal branch of z^β to $1 - f$, so $|\arg(1 - f)^\beta| \leq \beta\pi/2 < \pi/2$). Now fix β and $\varepsilon > 0$. For $\operatorname{Re} z \geq 0$ and $z = re^{i\theta} \neq 0$ ($|\theta| \leq \pi/2$)

$$\begin{aligned} (2) \quad \left| \exp\left(-\frac{\varepsilon}{z^\beta}\right) \exp \frac{k}{|z|^\alpha} \right| &= \exp(-\varepsilon r^{-\beta} \cos \beta\theta + kr^{-\alpha}) \\ &= \exp\{-r^{-\beta}(\varepsilon \cos \beta\theta - kr^{\beta-\alpha})\} \\ &\leq \exp(-cr^{-\beta}) \end{aligned}$$

for some $c > 0$ if r is sufficiently small, and this of course implies (2) is bounded on $\operatorname{Re} z \geq 0$. Thus

⁽¹⁾ The Choquet boundary consists of all points in the Šilov boundary having unique representing measures. In the metric case it coincides with the set of peak points.

$$(3) \quad \exp\left(-\frac{\varepsilon}{(1-f)^\beta}\right) \exp\left(\frac{k}{|1-f|^\alpha}\right)$$

is bounded on $\mathfrak{M}\setminus F$, whence

$$(4) \quad g \exp\left(-\frac{\varepsilon}{(1-f)^\beta}\right)$$

is bounded on $\mathfrak{M}\setminus F$ as the product of (1) and (3). But the exponential in (4), and thus (4) itself, is A -holomorphic and we can argue that by [1, Th, 4.8], (4) is bounded on $\mathfrak{M}\setminus F$ by its bound over $\partial\setminus F$, hence by $\sup|g(\partial\setminus F)|$ since the exponential is of modulus ≤ 1 . So for any φ in $\mathfrak{M}\setminus F$ we have

$$(5) \quad \left|g(\varphi) \exp\left(-\frac{\varepsilon}{(1-f(\varphi))^\beta}\right)\right| \leq \sup|g(\partial)\setminus F|,$$

and letting $\varepsilon \rightarrow 0$ yields the desired result.

Actually, once we have seen (4) is a bounded A -holomorphic function we should appeal directly to Rossi's local maximum modulus principle [4] to obtain (5). Indeed, extend (4) to all of \mathfrak{M} by setting it equal to zero on F ; since (3) tends to zero as we approach F (by (2)) we obtain a continuous function h on \mathfrak{M} . Now let B be the closed subalgebra of $C(\mathfrak{M})$ generated by h and A . To obtain (5) we need only see $\partial_B \subset \partial$ since then

$$|h(\varphi)| \leq \sup|h(\partial_B)| \leq \sup|h(\partial)| = \sup|h(\partial\setminus F)|,$$

because h vanishes on F , and this is (5).

We now argue exactly as in [1, 3.2]: if $\varphi \in \partial_B \cap (\mathfrak{M}\setminus\partial)$ we choose a neighborhood U_φ of φ in $\mathfrak{M}\setminus\partial$ on which h (and thus any element of B) is uniformly approximable by elements of A . Since $\varphi \in \partial_B$ we must have a φ' in U_φ and an h' in B with

$$(6) \quad |h'(\varphi')| > \sup|h'(\text{bdry } U_\varphi)|$$

and thus this holds for some approximating element h'' in A . But that violates the local maximum modulus principle, so $\partial_B \cap (\mathfrak{M}\setminus\partial) = \emptyset$, and $\partial_B \subset \partial$.

This argument yields a simple proof of Theorem 2. In that result, as is now apparent, we need only show the function

$$h = g \exp\left(\frac{-\varepsilon}{(1-f)^\beta}\right)$$

on $\mathfrak{M}\setminus\partial$ is bounded by its bound on V .

Now choose a deleted neighborhood W of F on which

$$|h| < \sup |h(V)| + \eta$$

(where $\eta > 0$), which is possible since $h \rightarrow 0$ as we approach F , exactly as before. Removing $(V \cup W)^-$ from the interior $\mathfrak{M} \setminus \partial$ we obtain an open subset U of $\mathfrak{M} \setminus \partial$ with $U^- \cap \partial = \emptyset$ so that $\text{bdry } U \subset (V \cup W)^-$. With B now the closed subalgebra of $C(U^-)$ generated by A and h we see that $\partial_B \subset \text{bdry } U$ by just the above application of local maximum modulus. Hence $\partial_B \subset (V \cup W)^-$, so that

$$\sup |h(\mathfrak{M} \setminus \partial)| \leq \sup |h(V \cup W)| \leq \sup |h(V)| + \eta;$$

since $\eta > 0$ is arbitrary, this shows h is bounded by its bound over V , as desired.

We can now proceed to the proof of Theorem 3, which involves some modifications in the arguments of [1, §4]. Let B_0 denote the uniformly closed algebra of bounded functions on \mathfrak{M} generated by g and A ; trivially \mathfrak{M} can be viewed as a subset of \mathfrak{M}_{B_0} and we let X denote the closure of \mathfrak{M} in \mathfrak{M}_{B_0} . X is a boundary for B_0 , so $B = B_0 \hat{=} X$ is a closed subalgebra of $C(X)$.

Since g and the elements of A are continuous when restricted to either ∂ or $\mathfrak{M} \setminus \partial$, the natural injection of each of these spaces into X is continuous, and of course one-to-one. In particular then the compact space ∂ is imbedded homeomorphically in X . But in fact the same is true of $\mathfrak{M} \setminus \partial$ since the map $\rho: X \rightarrow \mathfrak{M}$ dual to $A \rightarrow B$ clearly provides inverses for the injections $\partial \rightarrow X, \mathfrak{M} \setminus \partial \rightarrow X$. (Note that $\hat{f}(x) = f(\rho(x))$ for $f \in A, x \in X$.)

Now each of the sets $\mathfrak{M} \setminus \partial$ and $\partial \setminus (\mathfrak{M} \setminus \partial)^-$ is imbedded as an open subset of X . To see this note that each φ_0 in $\mathfrak{M} \setminus \partial$ has a compact neighborhood in \mathfrak{M} disjoint from ∂ of the form

$$U = \{\varphi \in \mathfrak{M}: |f_i(\varphi) - f_i(\varphi_0)| \leq \varepsilon, \quad i = 1, \dots, n\};$$

since $X = (\mathfrak{M} \setminus U)^- \cup U^- = (\mathfrak{M} \setminus U)^- \cup U, x \in X \setminus U$ implies $x \in (\mathfrak{M} \setminus U)^-$, and so $|\hat{f}_i(x) - f_i(\varphi_0)| \geq \varepsilon$ for some i , whence

$$\begin{aligned} W_{\varphi_0} &= \{\varphi \in \mathfrak{M}: |f_i(\varphi) - f_i(\varphi_0)| < \varepsilon/2, i = 1, \dots, n\} \\ &= \{x \in X: |\hat{f}_i(x) - f_i(\varphi_0)| < \varepsilon/2, i = 1, \dots, n\} \end{aligned}$$

is a neighborhood of φ_0 in X lying wholly within $\mathfrak{M} \setminus \partial$, so $\mathfrak{M} \setminus \partial$ is open in X as asserted. The same argument, starting from a compact neighborhood in ∂ disjoint from $(\mathfrak{M} \setminus \partial)^-$, yields a neighborhood W_{φ_0} of $\varphi_0 \in \partial \setminus (\mathfrak{M} \setminus \partial)^-$ in X lying wholly in $\partial \setminus (\mathfrak{M} \setminus \partial)^-$, so this set is also open in X . Moreover, the existence of W_{φ_0} shows ρ is one-to-one over $\mathfrak{M} \setminus \partial$ and $\partial \setminus (\mathfrak{M} \setminus \partial)^-$. For $\hat{f}_i(x) = \hat{f}_i(\rho(x)), x \in X, f_i \in A$, so $\rho(x) \in \mathfrak{M} \setminus \partial$ implies $x \in W_{\rho(x)} \subset \mathfrak{M} \setminus \partial$; similarly $\rho(x) \in \partial \setminus (\mathfrak{M} \setminus \partial)^-$ implies $x \in \partial \setminus (\mathfrak{M} \setminus \partial)^-$. So

$$\rho^{-1}(\mathfrak{M} \setminus \partial) = \mathfrak{M} \setminus \partial, \rho^{-1}(\partial \setminus (\mathfrak{M} \setminus \partial)^-) = \partial \setminus (\mathfrak{M} \setminus \partial)^-,$$

and thus ρ is clearly one-to-one over these sets.

Since $\mathfrak{M}\setminus\partial$ is open in X local maximum modulus applies to show $\partial_B \cap (\mathfrak{M}\setminus\partial) = \emptyset$ exactly as in [1, 3.2] or in our proof of Theorem 1: for any $\varphi \in \partial_B \cap (\mathfrak{M}\setminus\partial)$ has a neighborhood U_φ in $\mathfrak{M}\setminus\partial$ on which g (and so any element of B) is uniformly approximable by elements of A ; since $\mathfrak{M}\setminus\partial$ is open in X , U_φ is open in X and thus we find φ' in U_φ and h' in B satisfying (6) since $\varphi \in \partial_B$, and this contradicts local maximum modulus exactly as in the proof of Theorem 1. Thus

$$\partial_B \cap (\mathfrak{M}\setminus\partial) = \emptyset ,$$

and since $\rho^{-1}(\mathfrak{M}\setminus\partial) = \mathfrak{M}\setminus\partial$, we conclude that $\rho(\partial_B) \subset \partial$.

To complete our proof we need only see ρ is one-to-one on X : for then ρ is a homeomorphism of X with \mathfrak{M} (since $\rho(X) \subset \mathfrak{M}$ and $\rho(X) \supset (\mathfrak{M}\setminus\partial) \cup (\partial \setminus (\mathfrak{M}\setminus\partial)^-)$), and continuity of $g \circ \rho = \hat{g}$ on X implies that of g on $\mathfrak{M} = \rho(X)$.

We have already seen $\rho^{-1}(x) = \{x\}$ for x in $(\mathfrak{M}\setminus\partial) \cup (\partial \setminus (\mathfrak{M}\setminus\partial)^-)$, and for x in $\partial \setminus \bigcup K_n$ the assumed continuity of g at x implies $\rho^{-1}(x) = \{x\}$: for each h in B_0 is continuous at x , and so if $\rho(y) = x$ and the net $\{\varphi_\delta\}$ in \mathfrak{M} converges to y in X then $\rho(\varphi_\delta) = \varphi_\delta \rightarrow \rho(y) = x$ in \mathfrak{M} , whence $\hat{h}(y) = \lim \hat{h}(\varphi_\delta) = \hat{h}(x)$ for all h in B_0 , and $y = x$. Thus we need only see $\rho(y) = x$ for x in K_n implies $y = x$, and since we know this holds for x in $\partial \setminus (\mathfrak{M}\setminus\partial)^-$, we can assume $x \in (\mathfrak{M}\setminus\partial)^-$ as well.

So suppose $\rho(y) = x \in K_n \cap (\mathfrak{M}\setminus\partial)^-$. Since K_n lies in the Choquet boundary of A , only the unit point mass δ_x at x , among all probability measures on ∂ , can represent x on A . Thus if we knew $\partial_B = \partial$ then any probability measure μ on $\partial_B = \partial$ representing y on B would necessarily represent $\rho(y) = x$ on A , whence $\mu = \delta_x$ and $y = x$.

So we need only see $\partial_B \setminus \partial = \emptyset$ (since clearly $\partial \subset \partial_B$). As we saw, $\rho(\partial_B) \subset \partial$, and ρ is one-to-one over $(\partial \setminus \bigcup K_n) \cup (\partial \setminus (\mathfrak{M}\setminus\partial)^-)$ so that

$$\rho(\partial_B \setminus \partial) \subset (\mathfrak{M}\setminus\partial)^- \cap (\bigcup K_n) .$$

So by category if $\partial_B \setminus \partial \neq \emptyset$ one of the closed sets

$$E_n = \rho^{-1} [K_n \cap (\mathfrak{M}\setminus\partial)^-] \cap (\partial_B \setminus \partial)$$

in the locally compact space $\partial_B \setminus \partial$ has nonvoid interior in $\partial_B \setminus \partial$, hence in ∂_B . But $K_n = g_n^{-1}(0)$, $g_n \in A$ so that $y \in E_n$ lies in $\hat{g}_n^{-1}(0) = (g_n \circ \rho)^{-1}(0)$. In fact y lies in the topological boundary in X of $\hat{g}_n^{-1}(0)$. For

$$\rho(y) \in (\mathfrak{M}\setminus\partial)^-, y \notin \partial ,$$

and thus y has a neighborhood in X disjoint from ∂ , whence y lies in the closure in X of $\mathfrak{M}\setminus\partial$ (since $(\mathfrak{M}\setminus\partial) \cup \partial$ is dense in X). But $\hat{g}_n^{-1}(0) \cap (\mathfrak{M}\setminus\partial) = g_n^{-1}(0) \cap (\mathfrak{M}\setminus\partial) = \emptyset$, so that y lies in the topological

boundary of $\hat{g}_n^{-1}(0)$ as asserted.

Thus we have seen that E_n has nonvoid interior in ∂_B and lies in the topological boundary of $\hat{g}_n^{-1}(0)$ in X , which contradicts [1, 2.2]. Our assumption that $\partial_B \setminus \partial$ is nonvoid must therefore be false, and $\partial_B = \partial$ as desired, completing our proof.

Corollary 4 follows directly from the preceding. Indeed if we set

$$h = \begin{cases} g \exp\left(\frac{-\varepsilon}{(1-f)^\beta}\right) & \text{on } \mathfrak{M} \setminus K_1 \\ 0 & \text{on } K_1, \end{cases} \quad \alpha < \beta < 1,$$

then $h|_{\partial}$ is continuous and Theorem 3 implies $h \in C(\mathfrak{M})$. So h is bounded by its bound over ∂ , exactly as in the proof of Theorem 1, and so we see the same is true of g . Hence by Theorem 3, $g \in C(\mathfrak{M})$.

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