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**A SUM OF A CERTAIN DIVISOR FUNCTION FOR  
ARITHMETICAL SEMI-GROUPS**

E. M. HORADAM

## A SUM OF A CERTAIN DIVISOR FUNCTION FOR ARITHMETICAL SEMI-GROUPS

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Let  $\{b_n\}$  denote the set of elements of a free ordered arithmetical semi-group with multiplication and a known counting function. Using the corresponding terminology of arithmetic let  $b_n = d\delta$  and let  $\tau'(b_n)$  denote the number of divisors  $d$  of  $b_n$  where both  $d$  are  $\delta$  and square free. Then it is shown here that  $T(x)$  defined by

$$T(x) = \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \tau'(b_n) \sim Ax \log x$$

where  $A$  is a constant depending on  $b_u$ .

A more explicit definition of the semi-group is as follows. Suppose there is an infinite sequence  $\{p\}$  of real numbers, which we will call generalised primes, such that

$$1 < p_1 < p_2 < \dots$$

Form the set  $\{b\}$  of all  $p$ -products, i.e., products  $p_1^{v_1} p_2^{v_2} \dots$ , where  $v_1, v_2, \dots$  are integers  $\geq 0$  of which all but a finite number are 0. Call these numbers generalised integers and suppose that no two generalised integers are equal if their  $v$ 's are different. Then assume  $\{b\}$  may be arranged as an increasing sequence:

$$1 = b_1 < b_2 < \dots < b_n < \dots$$

We say  $d|b_n$  if  $d \in \{b\}$  and there exists  $\delta \in \{b\}$  such that  $d\delta = b_n$ ;  $d$  and  $\delta$  are then called complementary divisors of  $b_n$ . Let  $\tau'(b_n)$  be the number of divisors  $d$  of  $b_n$  where both  $d$  and its complementary divisor are square free. In fact

$$(1.1) \quad \tau'(b_n) = \sum_{\substack{d\delta=b_n \\ d \text{ square free} \\ \delta \text{ square free}}} 1.$$

This means that  $\tau'(b_n) = 0$  unless  $b_n$  is of the form  $\prod_{i,j} p_i p_j^2$ . Let  $x$  be any positive number and  $b_u$  any generalised integer. The sum to be evaluated,  $T(x)$  is defined by

$$(1.2) \quad T(x) = \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \tau'(b_n)$$

where  $(b_n, b_u)$  denotes the greatest common divisor of  $b_n$  and  $b_u$ . In

order to evaluate this sum a further assumption on the number of generalised integers less than or equal to  $x$  is required. Let  $[x]$  denote the number of generalised integers  $\leq x$ .

Assume

$$(1.3) \quad [x] = x + R(x), R(x) = o(x^\alpha) \text{ and } 0 < \alpha < 1.$$

Using (1.3) it will be shown that when  $b_u$  is square free

$$(1.4) \quad T(x) = Ax \log x + o\left(x \exp. \left\{ \frac{(\log b_u)^{1-\alpha}}{\log \log b_u} \right\}\right)$$

where

$$A = \prod_{p|b_u} \frac{p^2}{(p+1)^2} \prod_p \frac{(p^2-1)^2}{p^4}.$$

This sum is similar to that found by Gordon and Rogers in [2]. Also using the methods of [2] exactly analagous results for arithmetical semi-groups can be found to those shown by Gordon and Rogers. The only extra difficult result required is the prime number theorem for generalised integers. This is proved in [6] and is

$$(1.5) \quad \pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log^2 x}\right).$$

**2. Supplementary definitions and results.** Define the Möbius function  $\mu(b_n)$  for the semi-group as follows:  $\mu(b_n) = 0$  if  $b_n$  has a square factor  $\mu(b_n) = (-1)^k$ , where  $k$  denotes the number of prime divisors of  $b_n$  and  $b_n$  has no square factor;  $\mu(1) = 1$ . Let  $\phi(x, b_u)$  denote the number of generalised integers  $\leq x$  which are prime to  $b_u$ . Then it is proved in [3] that

$$(2.1) \quad \sum_{d|b_n} \mu(d) = \begin{cases} 0 & \text{when } b_n \neq 1 \\ 1 & \text{when } b_n = 1, \end{cases}$$

and in [4] that

$$(2.2) \quad \phi(x, b_u) = \sum_{d|b_u} \mu(d) \left[ \frac{x}{d} \right].$$

Hence using assumption (1.3) we have

$$(2.3) \quad \begin{aligned} \phi(x, b_u) &= x \sum_{d|b_u} \frac{\mu(d)}{d} + o\left(x^\alpha \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha}\right). \\ &= xf(b_u) + o(x^\alpha f_\alpha(b_u)) \text{ say.} \end{aligned}$$

Then as is shown in [3], and in any case as the functions are multiplicative

$$(2.4) \quad \begin{aligned} f(b_u) &= \sum_{d|b_u} \frac{\mu(d)}{d} = \prod_{p|b_u} \left(1 - \frac{1}{p}\right), \\ f_\alpha(b_u) &= \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha} = \prod_{p|b_u} \left(1 + \frac{1}{p^\alpha}\right). \end{aligned}$$

Define  $\zeta(s) = \sum_{n=1}^\infty b_n^{-s} (s > 1)$ . Then it is proved in [1], using an assumption equivalent to (1.3) that

$$\zeta(s) = \prod_{r=1}^\infty (1 - p_r^{-s})^{-1}.$$

Hence

$$\frac{1}{\zeta(s)} = \prod_{r=1}^\infty (1 - p_r^{-s}) = \sum_{n=1}^\infty \mu_n b_n^{-s}.$$

Abel's transformation, in the following form, will be used to give some necessary estimates. Suppose  $\{b_n\}$  and  $\{a_n\}$  are given with  $b_1 \leq b_2 \leq \dots, b_n \rightarrow \infty$ . Let  $A(x) = \sum_{b_n \leq x} a_n$ . Suppose  $\psi(x)$  has a continuous derivative  $\psi'(x)$  for all  $x$  involved. Then

$$\sum_{b_n \leq x} a_n \psi(b_n) = A(x)\psi(x) - \int_{b_1}^x A(u)\psi'(u)du.$$

Using (1.3) and this transformation, we obtain the following results.

$$(2.5) \quad \sum_{b_n \leq x} \frac{1}{b_n^\beta} = \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + o(x^{\alpha-\beta}), \quad \begin{cases} \beta \neq 1 \\ \beta \neq \alpha \end{cases},$$

and  $\gamma_\beta$  is a constant equal to  $\zeta(\beta)$  when  $\beta > 1$ .

$$(2.6) \quad \sum_{b_n \leq x} \frac{1}{b_n^\alpha} = \frac{x^{1-\alpha}}{1-\alpha} + o(\log x).$$

$$(2.7) \quad \sum_{b_n > x} \frac{1}{b_n^\beta} = \zeta(\beta) - \sum_{b_n \leq x} \frac{1}{b_n^\beta} = o(x^{1-\beta}) \quad \text{for } \beta > 1.$$

Again using (1.5) and Abel's transformation we obtain

$$(2.8) \quad \sum_{p \leq x} \frac{1}{p^\alpha} = \frac{x^{1-\alpha}}{(1-\alpha)\log x} + o\left(\frac{x^{1-\alpha}}{\log^2 x}\right)$$

$$(2.9) \quad \sum_{p \leq x} \log p = x + o(x/\log x).$$

Define

$$(2.10) \quad \lambda(b_u) = \sum_{\substack{b_n=1 \\ (b_n, b_u)=1}}^\infty \frac{\mu(b_n)}{b_n^2} = \prod_{p|b_u} \left(1 - \frac{1}{p^2}\right).$$

Then from (2.7) we have

$$(2.11) \quad \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \frac{\mu(b_n)}{b_n^2} = \lambda(b_u) + O(x^{-1}).$$

3. The  $Q$  function. Let

$$q_u(b_n) = \begin{cases} 1 & \text{if } b_n \text{ is square free and } (b_n, b_u) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$Q_u(x) = \sum_{b_n \leq x} q_u(b_n).$$

$$e(b_n) = \begin{cases} 1 & \text{if } b_n = 1 \\ 0 & \text{if } b_n \neq 1. \end{cases}$$

Then from (2.1)

$$q_u(b_n) = e((b_n, b_u)) \sum_{d^2 \delta = b_n} \mu(d).$$

This gives

$$\begin{aligned} Q_u(x) &= \sum_{b_n \leq x} e((b_n, b_u)) \sum_{d^2 \delta = b_n} \mu(d) \\ &= \sum_{\substack{d^2 \delta \leq x \\ (d, b_u) = (\delta, b_u) = 1}} \mu(d) \\ &= \sum_{\substack{d \leq \sqrt{x} \\ (d, b_u) = 1}} \mu(d) \phi\left(\frac{x}{d^2}, b_u\right) \\ &= \sum_{\substack{d \leq \sqrt{x} \\ (d, b_u) = 1}} \mu(d) \left\{ \frac{x}{d^2} f(b_u) + O\left(\frac{x^\alpha}{d^{2\alpha}} f(b_u)\right) \right\} \end{aligned}$$

from (2.3)

$$= x f(b_u) \{ \lambda(b_u) + O(x^{-1/2}) \} + O\left( x^\alpha f_\alpha(b_u) \left\{ \frac{x^{(1-2\alpha)/2}}{1-2\alpha} + \gamma_{2\alpha} \right\} \right)$$

from (2.11) and (2.5). Hence

$$(3.1) \quad Q_u(x) = x f(b_u) \lambda(b_u) + O(x^{1/2} f_\alpha(b_u)) + O(x^\alpha f_\alpha(b_u)).$$

4. The evaluation of the sum of the divisor function  $T(x)$ . Replacing in (1.2) the value for  $\tau'(b_n)$  defined in (1.1), we have the result that  $T(x)$  is the number of elements in the class satisfying  $d\delta = b_n$ ,  $\mu^2(d) = \mu^2(\delta) = 1$ , where  $b_n \leq x$ ,  $(b_n, b_u) = 1$ . This is the same class as that for which  $d\delta \leq x$ ,  $(d, b_u) = (\delta, b_u) = 1$  and  $\mu^2(d) = \mu^2(\delta) = 1$ . Rearranging the order of summation we have that  $T(x)$  is the number of elements in the class satisfying  $\delta \leq x/d$ ,  $(\delta, b_u) = 1$ ,  $\delta$  square free, where  $d \leq x$ ,  $(d, b_u) = 1$  and  $d$  is square free. Hence

$$\begin{aligned}
 T(x) &= \sum_{d \leq x} q_u(d) \sum_{\delta \leq x/d} q_u(\delta) \\
 &= \sum_{d \leq x} q_u(d) \left\{ \frac{x}{d} f(b_u) \lambda(b_u) + O\left(\frac{x^{1/2}}{d^{1/2}} f_\alpha(b_u)\right) + O\left(\frac{x^\alpha}{d^\alpha} f_\alpha(b_u)\right) \right\}
 \end{aligned}$$

from (3.1)

$$\begin{aligned}
 &= x f(b_u) \lambda(b_u) \sum_{d \leq x} \frac{q_u(d)}{d} + O\left(x^{1/2} f_\alpha(b_u) \sum_{d \leq x} \frac{1}{d^{1/2}}\right) \\
 &\qquad\qquad\qquad + O\left(x^\alpha f_\alpha(b_u) \sum_{d \leq x} \frac{1}{d^\alpha}\right) \\
 &= x f(b_u) \lambda(b_u) \sum_{d \leq x} \frac{q_u(d)}{d} + O(x f_\alpha(b_u))
 \end{aligned}$$

from (2.5) and (2.6).

Now from (3.1) and Abel's transformation we have

$$\begin{aligned}
 \sum_{d \leq x} \frac{q_u(d)}{d} &= f(b_u) \lambda(b_u) \log x + O(f_\alpha(b_u)) \\
 &\qquad\qquad\qquad + O(x^{-1/2} f_\alpha(b_u)) + O(x^{\alpha-1} f_\alpha(b_u)) .
 \end{aligned}$$

Substituting this result in the expression for  $T(x)$  we obtain

$$(4.1) \qquad T(x) = f^2(b_u) \lambda^2(b_u) x \log x + O(x f_\alpha(b_u)) .$$

From the definition in (2.4) we have

$$(4.2) \qquad f_\alpha(b_u) = \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha} \leq \sum_{d|b_u} \frac{1}{d^\alpha} \leq \sum_{d|b_u} 1 = O(b_u^\delta)$$

where  $\delta$  is any positive real number. This is proved in [5, Th. 5] and is true for all  $b_u$ . However, when  $b_u$  is square free we can obtain a better value for  $f_\alpha(b_u)$  by using the prime number theorem. Suppose  $b_u$  is square free and let  $b_u = p_{u1} p_{u2} \cdots p_{uk} \geq p_1 p_2 \cdots p_k$ . Then

$$(4.3) \qquad \log b_u \geq \sum_{p \leq p_k} \log p = p_k + O(p_k / \log p_k)$$

from (2.9). Hence

$$\begin{aligned}
 f_\alpha(b_u) &= \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha} = \prod_{p|b_u} \left(1 + \frac{1}{p^\alpha}\right) \leq \prod_{p \leq p_k} \left(1 + \frac{1}{p^\alpha}\right) \\
 &\leq \prod_{p \leq (1+o(1)) \log b_u} \left(1 + \frac{1}{p^\alpha}\right)
 \end{aligned}$$

from (4.3), and so

$$\log f_\alpha(b_u) \leq \sum_{p \leq (1+o(1)) \log b_u} \frac{1}{p^\alpha} (1 + o(1)) .$$

Then from (2.8)

$$(4.4) \quad f_\alpha(b_u) = 0 \left( \exp. \left\{ \frac{(\log b_u)^{1-\alpha}}{\log \log b_u} \right\} \right)$$

for  $b_u$  square free. Now from (2.4) and (2.10)

$$\begin{aligned} f^2(b_u)\lambda^2(b_u) &= \prod_{p|b_u} \left(1 - \frac{1}{p}\right)^2 \prod_{p|b_u} \left(1 - \frac{1}{p^2}\right)^2 \\ &= \prod_{p|b_u} \frac{p^2}{(p+1)^2} \prod_p \frac{(p^2-1)^2}{p^4} \\ &= A \text{ (say) .} \end{aligned}$$

Hence from (4.1), (4.2) and (4.4) we have

$$(4.5) \quad T(x) = Ax \log x + O(xb_u^\delta)$$

for all  $b_u$  and all positive real numbers  $\delta$  and

$$(4.6) \quad T(x) = Ax \log x + O \left( x \exp \left\{ \frac{(\log b_u)^{1-\alpha}}{\log \log b_u} \right\} \right)$$

for all square free  $b_u$ .

This is the result given for  $T(x)$  in (1.4). Since

$$\prod_p \left(1 - \frac{1}{p^2}\right)^2 = \frac{1}{\zeta^2(2)},$$

the value for  $A$  may also be written

$$A = \frac{1}{\zeta^2(2)} \prod_{p|b_u} \frac{p^2}{(p+1)^2}.$$

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#### REFERENCES

1. A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés*, Acta. Math. **68** (1937), 255-291.
2. G. Gordon and K. Rogers, *Sums of the divisor function*, Cana. J. Math. **16** (1964), 151-158.
3. E. M. Horadam, *Arithmetical functions of generalised primes*, Amer. Math. Monthly **68** (1961), 626-629.
4. ———, *The number of unitary divisors of a generalised integer*, Amer. Math. Monthly **71** (1964), 893-895.
5. ———, *The order of arithmetical functions of generalized integers*, Amer. Math. Monthly **70** (1963), 506-512.
6. B. Nyman, *A general prime number theorem*, Acta. Math. **81** (1949), 299-307.

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