

Pacific Journal of Mathematics

ON SOME HYPONORMAL OPERATORS

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Let H be a Hilbert space and T a hyponormal operator ($T^*T - TT^* \geq 0$). The first result is: if $(T^*)^p T^q$ is a completely continuous operator then T is normal.

Secondly, part we introduce the class of operators on a Banach space which satisfy the condition

$$\|x\| = 1 \quad \|Tx\|^2 \leq \|T^2x\|$$

and we prove the following:

1. $\gamma_T = \lim \|T^n\|^{1/n} = \|T\|$;
2. if T is defined on Hilbert space and is completely continuous then T is normal.

In what follows for this section we suppose that T is a hyponormal operator on Hilbert space H .

THEOREM 1.1. *If T is completely continuous then T is normal.*

This is known ([1], [2], [3]).

The main result of this section is as follows.

THEOREM 1.2. *If $T^{*p}T^q$ is completely continuous where p and q are positive integers then T is normal.*

LEMMA. Let $\|T\| = 1$. Then in the Hilbert space H there exists a sequence $\{x_n\}$, $\|x_n\| = 1$ such that

- (1) $\|T^*x_n\| \rightarrow 1$
- (2) $\|T^m x_n\| \rightarrow 1 \quad m = 1, 2, 3, \dots,$
- (3) $\|T^*Tx_n - x_n\| \rightarrow 0$
- (4) $\|TT^*x_n - x_n\| \rightarrow 0$
- (5) $\|T^*T^m x_n - T^{m-1}x_n\| \rightarrow 0 \quad m = 1, 2, 3, \dots$

Proof. We observe that (1) \Rightarrow (4) and (2) \Rightarrow (3). Thus it remains to prove (1), (2), and (5).

By definition there exists a sequence $\{x_n\}$, $\|x_n\| = 1$ such that

$$\|T^*x_n\| \rightarrow \|T^*\| = \|T\| = 1.$$

It is known [3] that for x , $\|x\| = 1$

$$\|Tx\|^2 \leq \|T^2x\|.$$

Since

$$\|T^*x_n\|^2 \leq \|Tx_n\|^2 \leq \|T^2x_n\| \leq 1$$

we have

$$\lim \|T^2x_n\| = 1.$$

If

$$\begin{aligned} \|T^{k-1}x_n\| &\rightarrow 1 \\ \|T^kx_n\| &\rightarrow 1 \end{aligned}$$

then

$$\lim \|T^{k+1}x_n\| = 1.$$

Now

$$\left\| T^2 \frac{T^{k-1}x_n}{\|T^{k-1}x_n\|} \right\| \geq \left\| T \frac{T^{k-1}x_n}{\|T^{k-1}x_n\|} \right\|^2$$

we have

$$\|T^{k+1}x_n\| \rightarrow 1.$$

By induction we have the relation (2).

For (5) we put

$$y_n(m) = T^*T^m x - T^{m-1}x_n$$

and

$$\delta_n(m) = \|y_n(m)\|^2.$$

We have

$$\begin{aligned} \delta_n(m) &= \|T^*T^m x_n\|^2 - 2\|T^m x_n\|^2 + \|T^{m-1}x_n\|^2 \\ &\leq \|T^m x_n\|^2 - 2\|T^m x_n\|^2 + \|T^{m-1}x_n\|^2 \\ &= \|T^{m-1}x_n\|^2 - \|T^m x_n\|^2. \end{aligned}$$

By (2) we obtain that $\delta_n(m) \rightarrow 0$ for every m . This proves the lemma.

Proof of the Theorem 1.2. Let p and q the integers such that $T^{*p}T^q$ is a completely continuous operator. By the lemma

$$T^*T^q x_n - T^{q-1}x_n \rightarrow 0$$

($\{x_n\}$ is the sequence of lemma). It is clear that $\{T^{*p-1}T^{q-1}x_n\}$ admits a subsequence which is convergent. Also, by the lemma and this

result we obtain a subsequence of $\{T^{*p-2}T^{q-2}x_n\}$ which is convergent. The process can be repeated and we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is convergent.

Let $x_0 = \lim x_{n_k}$. Thus

$$\begin{aligned} T^*Tx_0 &= x_0 \\ TT^*x_0 &= 0. \end{aligned}$$

The closed subspace $M_T = \{x, TT^*x\}$ is a nonzero subspace. By the Lemma 2 of [2] T has a approximate proper value

$$Ty_n - \lambda y_n \rightarrow 0.$$

The above arguments show that every sequence of approximate eigenvectors $\{y_n\}$ of T belonging to $\bar{\lambda}$ with $|\bar{\lambda}| = 1$ contains a convergent subsequence so that $\bar{\lambda}$ is an eigenvalue of T^* , hence λ is of T .

Let M be the smallest closed linear subspace which contains every proper subspace of T and $N = M^\perp$. It is known that N is invariant for T^* and thus $T^{*p}T^q$ is completely continuous on N . It is known that T_N is hyponormal. This shows that $N = \{0\}$ and $M = H$. The theorem is proved.

II. In this section we introduce a class of operators on any Banach space B .

DEFINITION 2.1. The operator T is said to be of class N if

$$x \in B, \|x\| = 1 \quad \|Tx\|^2 \leq \|T^2x\|.$$

LEMMA 2.1. Every hyponormal operator is of class N .

Proof.

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*Tx\| \leq \|T^2x\|.$$

It is clear by this lemma that these operators are extension of a class of hyponormal operators.

LEMMA 2.2. If T is of class N and

$$(1) \|T\| = 1, \quad (2) \|x_n\| \rightarrow 1, \quad (3) \|Tx_n\| \rightarrow 1.$$

Then $\|T^m x_n\| \rightarrow 1$ ($m = 1, 2, 3, \dots$).

Proof. This is easy consequence of the inequality

$$\|T^m x_n\| = \|T^2 \cdot T^{m-2} x_n\| \geq \frac{\|T^{m-1} x_n\|}{\|T^{m-2} x_n\|}.$$

THEOREM 2.1. *If T is of class N*

$$\|T\| = \lim \|T^n\|^{1/n} = \delta_T.$$

Proof. For every n , Lemma 2.2. leads to relation $\|T^n\| = \|T\|^n$ which gives Theorem 2.1.

COROLLARY 2.1. *A generalised nilpotent operator T of the class N is necessarily zero.*

LEMMA 2.3. *If T is of class N on a Hilbert space H and $\|T\| = 1$ then*

$$M_{T^*} = \{x, TT^*x\}$$

is invariant under T .

Proof. Let $x \in M_{T^*}$, $\|x\| = 1$. Then

$$\begin{aligned} \|T^*Tx - x\|^2 &= \|T^*Tx\|^2 - 2\|Tx\| + 1 \\ &= \|T^*Tx\|^2 - 2\|TT^*x\|^2 + 1 \\ &= \|T^*Tx\|^2 - 2\left\|T^2 \frac{T^*x}{\|T^*x\|}\right\|^2 \cdot \|T^*x\|^2 + 1 \\ &\leq \|T^*Tx\|^2 - 2\|TT^*x\|^4 \frac{1}{\|T^*x\|^2} + 1 \\ &\leq \|T^*Tx\|^2 - \frac{2}{\|T^*x\|^2} + 1 \leq 0. \end{aligned}$$

Thus $\|T^*Tx - x\| = 0$. It is clear that

$$Tx = TT^*(Tx) = T(T^*Tx)$$

which shows that $Tx \in M_{T^*}$.

We observe that T/M is an isometric operator.

THEOREM 2.2 *If T is of the class N on a Hilbert space and T^* is completely continuous for some $k \geq 1$ then T is normal.*

Proof. (for $\|T\| = 1$) From the completely continuous property of T^k it is clear that the subspace

$$M_{T^*} = \{x, TT^*x\}$$

is not $\{0\}$. Also M_{T^*} is finite dimensional because it is invariant under T^k which is isometric and completely continuous and M_{T^*} reduces T . We consider the subspace $M_{T^*}^\perp$ and continue in this way and obtain that T is normal.

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