ON SOME HYponORMAL OPERATORS

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Let $H$ be a Hilbert space and $T$ a hyponormal operator $(T^*T - TT^* \geq 0)$. The first result is: if $(T^*)^pT^q$ is a completely continuous operator then $T$ is normal.

Secondly, part we introduce the class of operators on a Banach space which satisfy the condition

$$||x|| = 1 \quad ||Tx||^2 \leq ||T^2x||$$

and we prove the following:

1. $\gamma_r = \lim ||T^n||^{1/n} = ||T||$;
2. if $T$ is defined on Hilbert space and is completely continuous then $T$ is normal.

In what follows for this section we suppose that $T$ is a hyponormal operator on Hilbert space $H$.

**Theorem 1.1.** If $T$ is completely continuous then $T$ is normal.

This is known ([1], [2], [3]).

The main result of this section is as follows.

**Theorem 1.2.** If $T^*pT^q$ is completely continuous where $p$ and $q$ are positive integers then $T$ is normal.

**Lemma.** Let $||T|| = 1$. Then in the Hilbert space $H$ there exists a sequence $\{x_n\}, ||x_n|| = 1$ such that

1. $||T^*x_n|| \to 1$
2. $||T^m x_n|| \to 1$ \quad $m = 1, 2, 3, \cdots$,
3. $||Tx_n - x_n|| \to 0$
4. $||TT^* x_n - x_n|| \to 0$
5. $||T^*T^m x_n - T^{m-1}x_n|| \to 0$ \quad $m = 1, 2, 3, \cdots$.

**Proof.** We observe that (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3). Thus it remains to prove (1), (2), and (5).

By definition there exists a sequence $\{x_n\}, ||x_n|| = 1$ such that

$$||T^*x_n|| \to ||T^*|| = ||T|| = 1.$$

It is known [3] that for $x, ||x|| = 1$
Since

$$|| T^*x_n ||^2 \leq || T^2x || .$$

we have

$$\lim || T^2x_n || = 1.$$

If

$$|| T^{k-1}x_n || \to 1$$
$$|| T^kx_n || \to 1$$

then

$$\lim || T^{k+1}x_n || = 1.$$

Now

$$\left| \left| T^2 \frac{T^{k-1}x_n}{|| T^{k-1}x_n ||} \right| \right|^2 \geq \left| \left| T \frac{T^{k-1}x_n}{|| T^{k-1}x_n ||} \right| \right|^2$$

we have

$$|| T^{k+1}x_n || \to 1.$$

By induction we have the relation (2).

For (5) we put

$$y_n(m) = T^*T^m x - T^{m-1}x_n$$

and

$$\delta_n(m) = || y_n(m) ||^2.$$

We have

$$\delta_n(m) = || T^*T^m x_n ||^2 - 2 || T^m x_n ||^2 + || T^{m-1}x_n ||^2 \leq || T^m x_n ||^2 - 2 || T^m x_n ||^2 + || T^{m-1}x_n ||^2$$

$$= || T^{m-1}x_n ||^2 - || T^m x_n ||^2.$$

By (2) we obtain that $\delta_n(m) \to 0$ for every $m$. This proves the lemma.

**Proof of the Theorem 1.2.** Let $p$ and $q$ the integers such that $T^pT^q$ is a completely continuous operator. By the lemma

$$T^*T^q x_n - T^{q-1}x_n \to 0$$

($\{x_n\}$ is the sequence of lemma). It is clear that $\{T^p-1T^{q-1}x_n\}$ admits a subsequence which is convergent. Also, by the lemma and this
result we obtain a subsequence of \( \{T^{*p-2}T^{q-2}x_n\} \) which is convergent. The process can be repeated and we obtain a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) which is convergent.

Let \( x_0 = \lim x_{n_k} \). Thus

\[
T^*Tx_0 = x_0 \quad \quad TT^*x_0 = 0 .
\]

The closed subspace \( M_x = \{x, TT^*x = x\} \) is a nonzero subspace. By the Lemma 2 of [2] \( T \) has a approximate proper value

\[
Ty_n - \lambda y_n \to 0 .
\]

The above arguments show that every sequence of approximate eigenvectors \( \{y_n\} \) of \( T \) belonging to \( \lambda \) with \( |\lambda| = 1 \) contains a convergent subsequence so that \( \lambda \) is an eigenvalue of \( T^* \), hence \( \lambda \) is of \( T \).

Let \( M \) be the smallest closed linear subspace which contains every proper subspace of \( T \) and \( N = M^\perp \). It is known that \( N \) is invariant for \( T^* \) and thus \( T^*T^q \) is completely continuous on \( N \). It is known that \( T_N \) is hyponormal. This shows that \( N = \{0\} \) and \( M = H \). The theorem is proved.

**II.** In this section we introduce a class of operators on any Banach space \( B \).

**Definition 2.1.** The operator \( T \) is said to be of class \( N \) if

\[
x \in B, \quad ||x|| = 1 \quad \quad ||Tx||^2 \leq ||T^2x|| .
\]

**Lemma 2.1.** Every hyponormal operator is of class \( N \).

**Proof.**

\[
||Tx||^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq ||T^*Tx|| \leq ||T^2x|| .
\]

It is clear by this lemma that these operators are extension of a class of hyponormal operators.

**Lemma 2.2.** If \( T \) is of class \( N \) and

\[
(1) \quad ||T|| = 1 , \quad (2) \quad ||x_n|| \to 1 , \quad (3) \quad ||Tx_n|| \to 1 .
\]

Then \( ||T^mx_n|| \to 1 \) (\( m = 1, 2, 3, \ldots \)).

**Proof.** This is easy consequence of the inequality

\[
||T^mx_n|| = ||T^zT^m-zx_n|| \geq \frac{||T^m-x_n||}{||T^m-zx_n||} .
\]
Theorem 2.1. If $T$ is of class $N$

$$||T|| = \lim ||T^*||^{1/n} = \delta_T.$$ 

Proof. For every $n$, Lemma 2.2. leads to relation $||T^*|| = ||T||^n$ which gives Theorem 2.1.

Corollary 2.1. A generalised nilpotent operator $T$ of the class $N$ is necessarily zero.

Lemma 2.3. If $T$ is of class $N$ on a Hilbert space $H$ and $||T|| = 1$ then

$$M_T = \{x, TT^* = x\}$$

is invariant under $T$.

Proof. Let $x \in M_T$, $||x|| = 1$. Then

$$||T^*Tx - x||^2 = ||T^*Tx||^2 - 2 ||Tx|| + 1$$

$$= ||T^*Tx||^2 - 2 ||TT^*x||^2 + 1$$

$$= ||T^*Tx||^2 - 2 \left( T^* \frac{T^*x}{||T^*x||} \right)^2 ||T^*x||^2 + 1$$

$$\leq ||T^*Tx||^2 - 2 ||TT^*x||^4 \frac{1}{||T^*x||^2} + 1$$

$$\leq ||T^*Tx||^2 - \frac{2}{||T^*x||^2} + 1 \leq 0.$$ 

Thus $||T^*Tx - x|| = 0$. It is clear that

$$Tx = TT^*(Tx) = T(T^*Tx)$$

which shows that $Tx \in M_T$.

We observe that $T/M$ is an isometric operator.

Theorem 2.2 If $T$ is of the class $N$ on a Hilbert space and $T^*$ is completely continuous for some $k \geq 1$ then $T$ is normal.

Proof. (for $||T|| = 1$) From the completely continuous property of $T^k$ it is clear that the subspace

$$M_{T^*} = \{x, TT^* = x\}$$

is not $\{0\}$. Also $M_{T^*}$ is finite dimensional because it is invariant under $T^k$ which is isometric and completely continuous and $M_{T^*}$ reduces $T$. We consider the subspace $M_{T^*}^\perp$, and continue in this way and obtain that $T$ is normal.
I am indebted to the referee for his suggestions.

REFERENCES


Received December 22, 1965.

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The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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