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V. ISTRĂŢESCU

Let H be a Hilbert space and T a hyponormal operator ($T^*T - TT^* \geqq 0$). The first result is: if $(T^*)^p T^q$ is a completely continuous operator then T is normal.

Secondly, part we introduce the class of operators on a Banach space which satisfy the condition

$$||x|| = 1 \qquad ||Tx||^2 \leqq ||T^2x||$$

and we prove the following:

1. $\gamma_T = \lim ||T^n||^{1/n} = ||T||$;
2. **if T is defined on Hilbert space and is completely continuous then T is normal.**

In what follows for this section we suppose that T is a hyponormal operator on Hilbert space H.

THEOREM 1.1. *If T is completely continuous then T is normal.*

This is known ([1], [2], [3]).

The main result of this section is as follows.

THEOREM 1.2. *If $T^{*p}T^q$ is completely continuous where p and q are positive integers then T is normal.*

LEMMA. Let $||T|| = 1$. Then in the Hilbert space H there exists a sequence $\{x_n\}$, $||x_n|| = 1$ such that

$$||T^*x_n|| \to 1 \tag{1}$$

$$||T^mx_n|| \to 1 \qquad m = 1, 2, 3, \cdots, \tag{2}$$

$$||T^*Tx_n - x_n|| \to 0 \tag{3}$$

$$||TT^*x_n - x_n|| \to 0 \tag{4}$$

$$||T^*T^mx_n - T^{m-1}x_n|| \to 0 \qquad m = 1, 2, 3, \cdots. \tag{5}$$

Proof. We observe that (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3). Thus it remains to prove (1), (2), and (5).

By definition there exists a sequence $\{x_n\}$, $||x_n|| = 1$ such that

$$||T^*x_n|| \to ||T^*|| = ||T|| = 1\,.$$

It is known [3] that for x, $||x|| = 1$

$$\| Tx \|^2 \leqq \| T^2 x \| .$$

Since

$$\| T^* x_n \|^2 \leqq \| Tx_n \|^2 \leqq \| T^2 x_n \| \leqq 1$$

we have

$$\lim \| T^2 x_n \| = 1 .$$

If

$$\| T^{k-1} x_n \| \to 1$$
$$\| T^k x_n \| \to 1$$

then

$$\lim \| T^{k+1} x_n \| = 1 .$$

Now

$$\left\| T^2 \frac{T^{k-1} x_n}{\| T^{k-1} x_n \|} \right\| \geqq \left\| T \frac{T^{k-1} x_n}{\| T^{k-1} x_n \|} \right\|^2$$

we have

$$\| T^{k+1} x_n \| \to 1 .$$

By induction we have the relation (2).

For (5) we put

$$y_n(m) = T^* T^m x - T^{m-1} x_n$$

and

$$\delta_n(m) = \| y_n(m) \|^2 .$$

We have

$$\begin{aligned} \delta_n(m) &= \| T^* T^m x_n \|^2 - 2 \| T^m x_n \|^2 + \| T^{m-1} x_n \|^2 \\ &\leqq \| T^m x_n \|^2 - 2 \| T^m x_n \|^2 + \| T^{m-1} x_n \|^2 \\ &= \| T^{m-1} x_n \|^2 - \| T^m x_n \|^2 . \end{aligned}$$

By (2) we obtain that $\delta_n(m) \to 0$ for every m. This proves the lemma.

Proof of the Theorem 1.2. Let p and q the integers such that $T^{*p} T^q$ is a completely continuous operator. By the lemma

$$T^* T^q x_n - T^{q-1} x_n \to 0$$

($\{x_n\}$ is the sequence of lemma). It is clear that $\{T^{*p-1} T^{q-1} x_n\}$ admits a subsequence which is convergent. Also, by the lemma and this result we obtain a subsequence of $\{T^{*p-2}T^{q-2}x_n\}$ which is convergent. The process can be repeated and we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is convergent.

Let $x_0 = \lim x_{n_k}$. Thus

$$T^*Tx_0 = x_0$$
$$TT^*x_0 = 0 .$$

The closed subspace $M_T = \{x,\ TT^*_x = x\}$ is a nonzero subspace. By the Lemma 2 of [2] T has a approximate proper value

$$Ty_n - \lambda y_n \to 0 .$$

The above arguments show that every sequence of approximate eigenvectors $\{y_n\}$ of T belonging to $\bar{\lambda}$ with $|\bar{\lambda}| = 1$ contains a convergent subsequence so that $\bar{\lambda}$ is an eigenvalue of T^*, hence λ is of T.

Let M be the smallest closed linear subspace which contains every proper subspace of T and $N = M^{\perp}$. It is known that N is invariant for T^* and thus $T^{*p}T^q$ is completely continuous on N. It is known that T_N is hyponormal. This shows that $N = \{0\}$ and $M = H$. The theorem is proved.

II. In this section we introduce a class of operators on any Banach space B.

DEFINITION 2.1. The operator T is said to be of class N if

$$x \in B,\ \|x\| = 1 \qquad \|Tx\|^2 \leqq \|T^2x\| .$$

LEMMA 2.1. *Every hyponormal operator is of class N.*

Proof.

$$\|Tx\|^2 = \langle Tx,\ Tx\rangle = \langle T^*Tx,\ x\rangle \leqq \|T^*Tx\| \leqq \|T^2x\| .$$

It is clear by this lemma that these operators are extension of a class of hyponormal operators.

LEMMA 2.2. *If T is of class N and*

$$(1)\quad \|T\| = 1 , \qquad (2)\quad \|x_n\| \to 1 , \qquad (3)\quad \|Tx_n\| \to 1 .$$

Then $\|T^mx_n\| \to 1$ $(m = 1, 2, 3, \cdots,)$.

Proof. This is easy consequence of the inequality

$$\|T^mx_n\| = \|T^2 \cdot T^{m-2}x_n\| \geqq \frac{\|T^{m-1}x_n\|}{\|T^{m-2}x_n\|} .$$

THEOREM 2.1. *If T is of class N*

$$\| T \| = \lim \| T^n \|^{1/n} = \delta_T .$$

Proof. For every n, Lemma 2.2. leads to relation $\| T^n \| = \| T \|^n$ which gives Theorem 2.1.

COROLLARY 2.1. *A generalised nilpotent operator T of the class N is necessarily zero.*

LEMMA 2.3. *If T is of class N on a Hilbert space H and $\| T \| = 1$ then*

$$M_{T^*} = \{x,\ TT^* = x\}$$

is invariant under T.

Proof. Let $x \in M_{T^*}$, $\| x \| = 1$. Then

$$\begin{aligned} \| T^* Tx - x \|^2 &= \| T^* Tx \|^2 - 2 \| Tx \| + 1 \\ &= \| T^* Tx \|^2 - 2 \| TTT^* x \|^2 + 1 \\ &= \| T^* Tx \|^2 - 2 \left\| T^2 \frac{T^* x}{\| T^* x \|} \right\|^2 \cdot \| T^* x \|^2 + 1 \\ &\leqq \| T^* Tx \|^2 - 2 \| TT^* x \|^4 \frac{1}{\| T^* x \|^2} + 1 \\ &\leqq \| T^* Tx \|^2 - \frac{2}{\| T^* x \|^2} + 1 \leqq 0 . \end{aligned}$$

Thus $\| T^* Tx - x \| = 0$. It is clear that

$$Tx = TT^*(Tx) = T(T^* Tx)$$

which shows that $Tx \in M_{T^*}$.

We observe that T/M is an isometric operator.

THEOREM 2.2 *If T is of the class N on a Hilbert space and T^* is completely continuous for some $k \geqq 1$ then T is normal.*

Proof. (for $\| T \| = 1$) From the completely continuous property of T^k it is clear that the subspace

$$M_{T^*} = \{x,\ TT^* = x\}$$

is not $\{0\}$. Also M_{T^*} is finite dimensional because it is invariant under T^k which is isometric and completely continuous and M_{T^*} reduces T. We consider the subspace $M_{T^*}^{\perp}$ and continue in this way and obtain that T is normal.

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