ON /-SIMPLICIAL CONVEXITY IN VECTOR SPACES

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The paper is concerned with a generalized type of convexity, which is called \( l \)-simplicial convexity. The name is derived from the simplex with \( l \) vertices, an \( l \)-simplicial convex set being the union of all \((i - 1)\)-simplices with vertices in another set, \( i \) varying between 1 and \( l \). The basic space is a linear one.

For convex sets the \( l \)-order (which is a natural number associated to an \( l \)-simplicial convex set) is a decreasing function of \( l \). Several inequalities between \( l \)- and \( k \)-orders are established. In doing this the case of a convex set and that of a non convex set are distinguished.

Besides these inequalities, the main result of the paper is the proof of non monotonicity of the \( l \)-order, given by an example in a 34-dimensional linear space.

Let us consider a real vector space and recall some notations and definitions, given in [2].

The convex cover (hull) of a set \( M \) is \( \mathcal{E}(M) \). We denote \( \mathcal{E}(\{x_1, \ldots, x_s\}) \) by \( \mathcal{E}(x_1, \ldots, x_s) \), \( x_1, \ldots, x_s \) being not necessarily distinct points in the space.

The operation \( \mathcal{S}_l \), called \( l \)-simplicial convex cover, and defined by

\[
\mathcal{S}_l(M) = \{ \bigcup \mathcal{E}(p_1, \ldots, p_i) : p_j \in M, 1 \leq j \leq l \}
\]

(for arbitrary set \( M \) and natural number \( l \geq 2 \)) will play a significant role throughout the paper. The operation \( \mathcal{S}_l \) is studied in [1], where is denoted by \( \text{con}_l \). It is easy to verify the following elementary properties of this operation:

1. \( \mathcal{S}_{mn} = \mathcal{S}_m \circ \mathcal{S}_n \),
2. \( \mathcal{S}_{mp} = \mathcal{S}_n \), and
3. \( \mathcal{S}_m(M) \subset \mathcal{E}(M) \),

for arbitrary \( m, n, p \geq 2 \) and \( M \). \( \mathcal{S}_l(M) \) is an increasing\(^1\) function of \( l \). It is also an increasing function of \( M \) with respect to the inclusion ordering. Let us denote \( \mathcal{S}_m(\{x_1, \ldots, x_s\}) \) by \( \mathcal{S}_m(x_1, \ldots, x_s) \).

A set \( K \) is said to be \( l \)-simplicial convex if there exists a subset \( M \) such that \( K = \mathcal{S}_l(M) \).

The \( l \)-order (\( l \)-simplicial convexity order) of an \( l \)-simplicial convex set \( K \) is

\[
\omega_l(K) = \sup_{M} \min \{ k : \mathcal{S}_l^k(M) = K \} .
\]

\(^1\) By increasing and decreasing functions we mean here not necessarily strictly increasing and strictly decreasing ones.
A set $K$ is said to be simplicial convex if there exists a number $l$ such that $K$ is $l$-simplicial convex.

The degree of a simplicial convex set $K$ is

$$
\delta(K) = \min \{l: K \text{ is } l\text{-simplicial convex}\}.
$$

The order (simplicial convexity order) of a simplicial convex set $K$ is

$$
\Omega(K) = \sup_{i} \omega_i(K).
$$

The power of a simplicial convex set $K$ of finite order is

$$
\Delta(K) = \min \{l: \Omega(K) = \omega_l(K)\}.
$$

It is proved in § 1 of both [1] and [2] that

$$
\mathcal{E}(M) = \mathcal{S}_{[\log_{l^*} l + 1]}(M)
$$
in an $n$-dimensional vector space.

2. Relations between $l$-simplicial and $k$-simplicial convexity orders. In this section some inequalities concerning $l$-order and $k$-order of a same set will be established.

**Theorem 1.** The $l$-order and $k$-order of a convex set $K$ satisfy the inequalities:

$$
\lfloor \log_{l^*}(l^*_{l-1})^1 + 1\rfloor \leq \omega_k(K) \leq \lfloor \log_{l^*}(l^*_{l-1})^1 - 1\rfloor + 1.
$$

**Proof.** Consider a set $M$ such that

$$
\mathcal{S}_{l^*}(M) = K.
$$

If $k < l^*$,

$$
K = \mathcal{S}_{l^*}(M) \subset \mathcal{S}_{k^*}(M).
$$

Since $K$ is convex the inverse inclusion holds too and $K = \mathcal{S}_{k^*}(M)$.

Let $x \in K$. Since

$$
\mathcal{S}_{l^*_{l-1}}(M) = K,
$$
x belongs to a simplex with vertices $x_1, \ldots, x_s \in M (s \leq l^*_{l-1})$. There exists a linear manifold of dimension $l^*_{l-1} - 1$ containing $\mathcal{E}(x_1, \ldots, x_s)$. Following the last remark of § 1,

$$
\mathcal{E}(x_1, \ldots, x_s) = \mathcal{S}_{[\log_{l^*}[l^*_{l-1}]^1]}(x_1, \ldots, x_s) \subset \mathcal{S}_{[\log_{l^*}[l^*_{l-1}]^1]}(M),
$$
hence
The inverse inclusion holds owing to the convexity of $K$, whence

$$\omega^k(K) = K.$$  

Thus

$$\omega^k(K) \leq \log_k (l^{\omega^k(K)} - 1) + 1.$$  

The symmetry in $l$ and $k$ gives

$$\omega_l(K) \leq [\log_l (k^{\omega_l(K)} - 1)] + 1 \leq \log_l (k^{\omega_l(K)} - 1) + 1,$$

i.e.

$$l^{\omega_l(K) - 1} \leq k^{\omega_l(K)} - 1.$$  

It follows that

$$[\log_k (l^{\omega_l(K) - 1} + 1)] \leq \log_k (l^{\omega_l(K) - 1} + 1) \leq \omega_k(K)$$

and both inequalities are obtained.

In [2], we have established that in general, $k$-simplicial convexity does not imply $l$-simplicial convexity for $l < k$. However, if $k$ is a power of $l$ this implication holds.

**Theorem 2.** For all natural numbers $k, l, q \geq 2$ satisfying $k = l^q$, non convexity and $k$-simplicial convexity imply $l$-simplicial convexity; also the $k$-simplicial and $l$-simplicial convexity orders verify the inequalities

$$q\omega_k \leq \omega_l \leq q\omega_k + q - 1.$$  

**Proof.** Let $M$ be such that

$$\mathcal{S}_k^{w^k(C)}(M) = C,$$

where $C$ is a non convex, $k$-simplicial convex set. Then

$$\mathcal{S}_l^{\omega_k(C)} = C,$$

whence $C$ is $l$-simplicial convex and

$$\omega_l(C) \geq q\omega_k(C).$$

To prove the other inequality, suppose that there is a set $M'$ such that

$$\mathcal{S}_l^{w^l(M')} = C$$

with
Then, either \( m = q \omega_k(C) + q \) and

\[
\mathcal{S}_{k}^{\omega_k(C)+1}(M') = C,
\]

or \( m > q \omega_k(C) + q \) and

\[
\mathcal{S}_{k}^{\omega_k(C)+1}(\mathcal{S}_{m-q \omega_k(C)-q}(M')) = C,
\]

both impossible. Following Theorem 5 of [2],

\[
\omega_l(C) = \sup \{ m : \mathcal{S}_l^m(M) = C \}.
\]

Hence

\[
\omega_l(C) \leq q \omega_k(C) + q - 1,
\]

which concludes the theorem.

A different inequality is obtained, if \( k = l^r \), for a convex set. We have, indeed, by Theorem 1, \( \omega_k \leq \log_k (l^\omega_i - 1) + 1 \), i.e.

\[
k^{\omega_k-1} \leq l^\omega_i - 1
\]

and \( \omega_k \geq \log_k (l^\omega_i - 1) + 1 \), i.e.

\[
k^{\omega_k} \geq l^\omega_i - 1.
\]

Hence

\[
l^{\omega_k - 1} \leq l^\omega_i - 1,
\]

whence

\[
q(\omega_k - 1) < \omega_l
\]

and

\[
l^{\omega_k} \geq l^\omega_i - 1,
\]

whence

\[
q \omega_k > \omega_l - 1.
\]

Therefore

\[
q \omega_k - q + 1 \leq \omega_l \leq q \omega_k.
\]

These inequalities and those of Theorem 2 show that, for arbitrary sets, \( k \)-simplicial convexity implies \( l \)-simplicial convexity (for \( k = l^r \)) and

\[
q \omega_k - q + 1 \leq \omega_l \leq q \omega_k + q - 1.
\]
3. **Monotonicity of \( \omega_l(K) \) for convex \( K \).** It is proved in §4 of [2] that \( \omega_k \preceq \omega_l \) if \( k \) is a multiple of \( l \). Moreover, we shall prove that for convex \( K \), \( \omega_k(K) \preceq \omega_l(K) \) if \( k \geq l \).

**Theorem 3.** For a convex set \( K \), the \( l \)-simplicial convexity order is decreasing on \( l \) and \( \Omega(K) = \omega_2(K) \).

**Proof.** Prove that \( \omega_i(K) \) is a decreasing function of \( i \). Suppose, on the contrary, that \( i > j \) and

\[
\omega_i(K) > \omega_j(K).
\]

It follows that

\[
\omega_i(K) - 1 \geq \omega_j(K)
\]

and

\[
\omega_i(K)^{-1} > \omega_j(K)^{-1} \geq \omega_j(K).
\]

But, from Theorem 1,

\[
\omega_i(K) \leq \log_i(\omega_j(K)^{-1} - 1) + 1,
\]

whence

\[
\omega_i(K)^{-1} \leq \omega_j(K)^{-1} - 1 < \omega_j(K).
\]

The contradiction shows that \( \omega_i(K) \) is decreasing of \( l \), for convex \( K \). Since \( K \) is 2-simplicial convex,

\[
\Omega(K) = \omega_{\delta(K)}(K) = \omega_2(K).
\]

4. **Non monotonicity of \( \omega_l \).** It may be conjectured that \( \omega_l \) is in general a decreasing function of \( l \), i.e., \( \omega_l(C) \) is also decreasing for non convex \( C \). Then the inequality of Theorem 11 would be trivially implied by Theorem 6, both of [2], and \( \Omega \) and \( \delta \) would equal respectively \( \omega_2 \) and \( \delta \).

On the other hand one can believe that Theorem 2 can be obtained from two more general inequalities, for non convex sets, like

\[
(*\) \quad [\log_l(l^{\omega_l} + 1) - 1 \leq \omega_k \leq [\log_l(l^{\omega_l} - 1)]
\]

in the same way as, for convex sets,

\[
q \omega_k - q + 1 \leq \omega_i \leq q \omega_k \quad (k = l^q)
\]

is implied by Theorem 1.

It should be noted that each inequality of (\( *\)) would imply that \( \omega_i \) is decreasing. If, on the contrary \( i < j \) and \( \omega_i < \omega_j \), then
which contradicts the two inequalities
\[ j^{a_j} \leq i^{w_i+1} - 1 \]
and
\[ \omega_j \leq [\log_j (i^{w_i+1} - 1)] . \]
Also,
\[ i^{w_i+1} < j^{a_j} \]
contradicts the two inequalities
\[ j^{a_j} + 1 \leq i^{w_i+1} \]
and
\[ [\log_i (j^{a_j} + 1)] - 1 \leq \omega_i . \]

The conjecture that \( \omega_i \) is, in general, a decreasing function of \( l \) (and with it also relation (†)) is disproved by the following counter-
example.

**PROPOSITION.** Let \( \mathcal{Y} \) be a 34-dimensional real vector space and \( C \) be the union of the subsimplexes with 18 vertices of a simplex with 35 vertices in \( \mathcal{Y} \). Then \( C \) is both 2-simplicial convex and 3-simplicial convex, \( \omega_2(C) = 1 \) and \( \omega_3(C) = 2 \). Also \( \Omega(C) = 2 \) and \( \Delta(C) = 3 \).

**Note.** This proposition and its proof, are the simplest that we could find to provide our counter-example. We ask for simpler ones. In fact, we have found such a simpler example in a vector space of smaller dimension, but the proof was much more complicated. However it should be interesting to find the smallest dimension of the space in which such an example can be found, even if its proof is difficult.

**Proof.** Let \( S_j^i (i = 1, \ldots, (j^2)) \) be the subsimplexes with \( j \) vertices \( (j = 2, \ldots, 34) \) of the given simplex \( S \). Thus

\[ C = \bigcup S_i^j . \]

(1) The 2-simplicial and 3-simplicial convexity of \( C \) follow from

\[ C = \mathcal{S} (\bigcup S_i^j) = \mathcal{S} (\bigcup S_i^j) . \]

(2) Prove that \( \omega_2(C) = 1 \). Suppose that \( \mathcal{S}^2(M) = C \).

(α) First, we shall prove that
Suppose there exists
\[ y \in \mathcal{S}_2(M) - \bigcup_i S_i^j; \]
then \( y \) belongs to the interior of a simplex \( S_i^{18} \) with \( 10 \leq j \leq 18 \).

(a) If \( 10 \leq j \leq 17 \), let \( S_i^{18-j} \) be the simplex with \( 35 - j \) vertices disjoint from \( S_j^{18} \). If \( \mathcal{S}_2(M) \) contains also a point \( z \) in the interior of a subsimplex \( S_i^{l-j} \) of \( S_i^{18-j} \), with \( l \geq 19 \), then \( \varepsilon(y, z) \) intersects the interior of \( \mathcal{G}(S_j^{18} \cup S_i^{l-j}) \), which is impossible; hence

\[
\mathcal{S}_2(M) \cap S_i^{18-j} \subset \left\{ \bigcup S_i^{18-j} : S_i^{18-j} \subset S_i^{18-j} \right\} \subset \left\{ \bigcup S_i^j : S_i^j \subset S_i^{18-j} \right\}
\]

for \( 10 \leq j \leq 16 \) or

\[
\mathcal{S}_2(M) \cap S_i^{18} \subset \{ x_1, \ldots, x_{35} \} \cap S_i^{18} \subset \{ \bigcup S_i^j : S_i^j \subset S_i^{18} \}
\]

and if

\[
S_i^{18} \subset S_i^{18-j},
\]
then

\[
\mathcal{S}_2(M) \cap S_i^{18} \subset \{ \bigcup S_i^j : S_i^j \subset S_i^{18-j} \} \cap S_i^{18} = \{ \bigcup S_i^j : S_i^j \subset S_i^{18} \}.
\]

It follows that

\[
\mathcal{S}_2(M) \cap S_i^{18} \subset \{ \bigcup S_i^j : S_i^j \subset S_i^{18} \} \neq S_i^{18},
\]
whence

\[
\mathcal{S}_2(M) \nsubseteq S_i^{18},
\]
absurd.

(b) If \( j = 18 \), then the segment joining \( y \) with a vertex \( x_\alpha \) of \( S \) that does not belong to \( S_i^{18} \), meets the interior of \( \mathcal{G}(S_i^{18} \cup \{ x_\alpha \}) \), absurd again.

(\beta) Now, prove that

\[ M \nsubseteq \bigcup_i S_i^i. \]
Suppose that \( M \subset \bigcup_i S_i^i \). Then

\[
\mathcal{S}_2(M) \subset \bigcup S_i^{18},
\]
whence

\[
\mathcal{S}_2(M) \neq \bigcup S_i^{18},
\]
impossible. Hence there exists \( x \in M - \bigcup_i S_i^i \).
(a) If $x$ is an interior point of a simplex $S^i_j$ with $5 \leq j \leq 8$, then, following (a), $M$ does not intersect the interior of any subsimplex $S^i_j(l \geq 5)$ of
\[ S^i_{35-j} = \mathcal{E}(\{x_1, \ldots, x_{38}\} - S^i_j). \]
Hence, if $S^i_{16} \subset S^i_{35-j}$, then
\[ \mathcal{E}^2(M) \cap S^i_{16} \subset \mathcal{E}^2(\{ \cup S^i_i : S^i_i \subset S^i_{18} \} = \{ \cup S^i_{16} : S^i_{16} \subset S^i_{18} \} \neq S^i_{16}, \]
whence $\mathcal{E}^2(M) \not\supset S^i_{16}$, absurd.

(b) If $x$ is an interior point of the simplex $S^i_{9}$ then, for $x_r \in S^i_9$, $\mathcal{E}(x, x_r)$ intersects the interior of $\mathcal{E}(S^i_9 \cup \{x_r\})$, absurd.

Combining (a) and (b) we obtain that $M \not\subset \bigcup_i S^i_i$ and $\mathcal{E}(M) \subset \bigcup_i S^i_i$, contradicting one other. Therefore any $M \subset \mathcal{V}$ such that $\mathcal{E}(M) = C$ does not exist. Also, $\mathcal{E}^k(M) \neq C$ for $k \geq 3$, because $\mathcal{E}^k(\mathcal{E}^{k-2}(M)) \neq C$.

Hence $\omega_4(C) = 1$.

(3) Prove that $\omega_3(C) = 2$. Suppose there exist a subset $M \subset \mathcal{V}$ and a number $k \geq 3$ such that $\mathcal{E}^k(M) = C$. Then $M$ must contain the vertices $x_1, \ldots, x_{38}$ of $S$. But
\[ \mathcal{E}^k(M) \supset \mathcal{E}^k(M) \supset \mathcal{E}^k(\{x_1, \ldots, x_{38}\}) = \bigcup_i S^i_{27}, \]
whence $\mathcal{E}^k(M) \neq C$, absurd. Because
\[ \mathcal{E}^2\left( \bigcup_i S^i_i \right) = C, \]
$\omega_3(C) = 2$.

(4) Prove that $\Omega(C) = 2$ and $\Delta(C) = 3$. If $C$ would be 4-simplicial convex, then $2\omega_4(C) \leq 1$, following Theorem 2, which is not possible. If $C$ is $l$-simplicial convex, with $l \geq 5$, then suppose that $M$ and $k \geq 2$ satisfy $\mathcal{E}^4(M) = C$. Then
\[ \mathcal{E}^4(M) \supset \mathcal{E}^4(M) \supset \mathcal{E}^4(\{x_1, \ldots, x_{38}\}) = \bigcup_i S^i_{25}, \]
as absurd. Hence $\omega_3(C) = 1$.

Thus $\Omega(C) = \omega_3(C) = 2$ and $\Delta(C) = 3$. The proof is complete.

5. Concluding remarks. Most of the results of [2] and of the present paper are purely algebraic and hence valid in vector spaces over arbitrary ordered fields.

Besides the operation $\mathcal{E}$, the function $\omega_i$ whose study was begun in § 2 of [2] and continued here is of special interest in the investigation of $l$-simplicial convexity. In §§ 3-4 of [2] we established some elementary facts concerning degree, order, and power. In order to initiate a systematic study of simplicial convexity more information about these three functions should be obtained.
REFERENCES


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