A topological semigroup $S$ is a Hausdorff space together with a continuous associative multiplication on $S$. A semigroup $S$ is said to be uniquely divisible if each element of $S$ has unique roots of each positive integral order in $S$. The present paper concerns uniquely divisible semigroups on the two-cell.

The main result of this paper is a statement of equivalent conditions for a commutative uniquely divisible semigroup on the two-cell to be the continuous homomorphic image of the cartesian product of two threads. This result is applied to determine the structure of commutative uniquely divisible semigroups on the two-cell whose idempotent set consists of a zero and an identity.

A $U$-semigroup is a semigroup which is isomorphic (topologically isomorphic) to the real unit interval $[0, 1]$ under usual multiplication. A thread is a semigroup on an arc such that one endpoint is a zero and the other endpoint is an identity.

For a semigroup $S$, $E(S)$ denotes the set of all idempotent elements of $S$. The statement "$E(S) = \{0, 1\}$" means that the only idempotents of $S$ are a zero (0) and an identity (1).

Throughout this paper $N$ denotes the set of all positive integers and $R$ denotes the set of all positive rational numbers. Hereafter the statement "$S$ is an UDS" means that $S$ is an uniquely divisible topological semigroup.

If $S$ is an UDS, $x \in S$, and $n \in N$, then $x^{1/n}$ denotes the unique $n$th root of $x$ in $S$. If $r \in R$, $r = m/n; m, n \in N$, and $x \in S$, then $x^r = (x^{1/n})^m$. It is not difficult to show that $x^r$ is unique for each $r \in R$. Define $[x] = \{x^r; r \in R\}^*$ (closure in $S$).

2. Preliminary results.

Theorem 2.1. Let $S$ be a compact UDS such that each subgroup of $S$ is totally disconnected. Then, for each $x \in S \setminus E(S)$, $[x]$ is a $U$-semigroup.

Proof. Let $H$ denote the maximal subgroup of $[x]$ containing the identity ($e$) of $[x]$, and let $K$ denote the kernel (minimal ideal) of $[x]$. Then $H$ and $K$ are connected subgroups of $S$. Hence $H = \{e\}$ and $K = \{f\}$, where $f$ is the identity of $K$.

There exists a continuous one-to-one homomorphism $\sigma$ from the
additive nonnegative real numbers $\bar{R}$ into $[x]$ such that $[x] = H\sigma(\bar{R})^*$ (closure in $[x]$) [4, Theorem 3.1]. Since $H = \{e\}$, $[x] = \sigma(\bar{R})$. Note that $\sigma(\bar{R})^*\sigma(\bar{R}) = \{f\}$ [4, Theorem 3.1].

Let $I = [0,1]$ under usual multiplication. Define $\psi: [x] \rightarrow I$ by $\psi(f) = 0$ and $\psi(p) = \exp(-\sigma^{-1}(p))$ if $p \neq f$. Then $\psi$ is an isomorphism of $[x]$ onto $I$.

**Corollary 2.2.** Let $S$ be a compact semigroup such that each subgroup of $S$ is totally disconnected. Then $S$ is an UDS if and only if each point of $S\setminus E(S)$ lies on an unique $U$-semigroup in $S$.

**Corollary 2.3.** Let $S$ be a semigroup on the two-cell. Then $S$ is an UDS if and only if each point of $S\setminus E(S)$ lies on an unique $U$-semigroup in $S$.

3. Uniquely divisible semigroups on the two-cell. Throughout this section $S$ denotes an UDS with identity (1) on the two-cell and $B$ denotes the boundary of $S$. Note that $1 \in B$ [10]. If $S$ has a zero $(0)$ and $0 \in B$, then $B_1$ and $B_2$ denote the boundary arcs from 0 to 1 in $S$. Thus $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \{0,1\}$.

**Lemma 3.1.** If $S$ has a zero $(0)$ and each point of $E(S)$ lies on a thread in $S$ containing 1, then each point of $S$ lies on a thread in $S$ from 0 to 1.

*Proof.* Since $0 \in E(S)$, there exists a thread $T$ from 0 to 1 in $S$. Let $e \in E(S)$. Then there exists a thread $T_e$ from $e$ to 1 in $S$. Now $eT$ is a thread from 0 to $e$ in $S$. Thus $eT \cup T_e$ contains a thread $T(e)$ from 0 to 1 in $S$ such that $e \in T(e)$. Hence, if $e \in E(S)$, then $e$ lies on a thread $T(e)$ from 0 to 1 in $S$.

Let $x \in S\setminus E(S)$. Then, by Corollary 2.3, $x$ lies on an unique $U$-semigroup $I$ in $S$. Let $z$ denote the zero of $I$ and $u$ the identity of $I$. Since $z, u \in E(S)$, there exists threads $T(z)$ and $T(u)$ from 0 to 1 in $S$ such that $z \in T(z)$ and $u \in T(u)$. Thus $T(z) \cup I \cup T(u)$ contains a thread $T^i$ from 0 to 1 in $S$ such that $x \in T^i$.

**Lemma 3.2.** If $E(S) = \{0,1\}$, then $0 \in B$.

*Proof.* Suppose $0 \notin B$. Let $x \in B\setminus E(S)$. Then $B\setminus [x] \neq \emptyset$. Let $p \in B\setminus [x]$. Since $[x] \cap B$ is closed, there exists a point $y$ in the arc from $p$ to $x$ on $B$ which does not contain 1. Then $[y]$ must meet $[p]$ or $[x]$ in a point $q$ not in $E(S)$. Thus $q$ lies on two distinct $U$-semigroups in $S$. This is a contradiction to Corollary 2.3. Hence $0 \in B$. 
Lemma 3.3. Suppose $S$ has zero $(0)$ and $0 \in B$. If each of $B_1$ and $B_2$ is a thread, then $S = B_1 B_2 = B_2 B_1$.

Proof. Now $1 \in B_1 \cap B_2$. Hence $B \subset B_1 B_2$. Define $\varphi : B_1 B_2 \to S$ by $\varphi((b,b_2,1)) = b b_2 b$. Then $\varphi$ is continuous, $\varphi((b,b_2,0)) = 0$, and $\varphi((b,b_2,1)) = b b_2$. Hence $B_1 B_2$ is contractible, and thus $S = B_1 B_2$. Similarly, $S = B_2 B_1$.

Lemma 3.4. Suppose $S$ has a zero $(0)$ and $0 \in B$. If each point of $S$ lies on a thread from $0$ to $1$ in $S$, then each of $B_1$ and $B_2$ is a thread.

Proof. Let $x$ and $y$ be distinct points of $B_1 \setminus \{0,1\}$ such that $y$ separates $x$ from $1$ on $B_1$. Suppose $[x] \neq [y]$. Let $T_1$ and $T_2$ denote threads from $0$ to $1$ in $S$ containing $x$ and $y$ respectively. Then, since $y$ separates $x$ from $1$ on $B_1$, $T_1 \cap T_2$ contains an idempotent $f$ such that $xf = x$ and $fy = f$. Hence $xy = (xf)y = x(fy) = xf = x$. Thus, if $y$ separates $x$ from $1$ on $B_1$ and $[x] \neq [y]$, then $xy = x$.

If $B_1 \setminus E(S) = \emptyset$, then the fact that $B_1$ is a thread follows from the preceding paragraph. Suppose $B_1 \setminus E(S) \neq \emptyset$. Let $z \in B_1 \setminus E(S)$. Then there exists a $U$-semigroup $I$ in $S$ such that $z \in I$. Let $a$ be the zero of $I$ and $b$ the identity of $I$. Let $M$ be the component of $I \cap B_1$ containing $z$, $h = \inf M$, and $g = \sup M$ in the cut-point ordering ($<$) of $B_1$ from $0$ to $1$. Since $h = \inf M$, there exists a sequence $\{h_n\}$ of points of $B_1 \setminus I$ such that $h_n < h$ for each $n \in N$ and $h_n \uparrow h$. Thus, by the preceding paragraph, $h_n h = h_n$ for each $n \in N$. Since multiplication is continuous in $S$, $h_n h \to h^2$. Hence $h = h^2$. Since $h \in I$, $a = h$. Similarly, $g = b$, and hence $I \subset B_1$. Thus $B_2$ is a thread. Similarly, $B_2$ is a thread. This completes the proof of Lemma 3.4.

A commutative UDS $S$ can be considered to be a generalization of a semilattice (a commutative idempotent semigroup). Indeed, if $S = E(S)$, then $S$ is a semilattice. Consequently, Theorem 3.5 is a generalization of Theorem 3 in [1].

If $S$ is commutative, then the kernel $K$ (the minimal ideal) of $S$ is a compact connected group. Hence $K$ is either the circle group $C$ or a point. It is not difficult to show that $K$ is uniquely divisible. Thus, since $C$ is not uniquely divisible, $K$ is a point. Hence, if $S$ is commutative, then $S$ has a zero $(0)$.

Theorem 3.5. If $S$ is commutative and $0 \in B$, then these are equivalent:

(i) each point of $E(S)$ lies on a thread in $S$ containing $1$;
(ii) each point of $S$ lies on a thread from $0$ to $1$ in $S$;
(iii) each of $B_1$ and $B_2$ is a thread;
(iv) \( S \) is the continuous homomorphic image of the cartesian product of two threads.

Proof. (i) implies (ii). [Lemma 3.1].

(ii) implies (iii). [Lemma 3.4].

(iii) implies (iv). By Lemma 3.3, \( S = B_1B_2 \).

Define \( \psi : B_1 \times B_2 \to S \) by \( \psi((b_1, b_2)) = b_1b_2 \). Then \( \psi \) is a continuous homomorphism onto \( S \).

(iv) implies (i). Let \( I_1 \) and \( I_2 \) be threads and \( \varphi \) a continuous homomorphism of \( I_1 \times I_2 \) onto \( S \). Let \( g \in E(S) \) and \( p \in \varphi^{-1}(g) \). Then there exists a thread from \((0, 0)\) to \((1, 1)\) in \( I_1 \times I_2 \) containing \( p \).

Hence, by Theorem 2 of [3], \( \varphi(T) \) is a thread in \( S \) containing \( g \) and 1.

**Corollary 3.6.** If \( S \) is commutative and \( E(S) = \{0, 1\} \), then \( S \) is isomorphic to \((I_1 \times I_2)/J \), where \( I = [0, 1] \) is a U-semigroup and \( J \) is the ideal \( \{(x, y): x = 0 \text{ or } y = 0\} \).

Proof. By Lemma 3.2, \( 0 \in B \). By Theorem 1 in [7], there exists a thread from 0 to 1 in \( S \). Therefore, by Theorem 3.5, each of \( B_1 \) and \( B_2 \) is a thread, and thus are U-semigroups. The map \( \psi : B_1 \times B_2 \to S \) defined by \( \psi((b_1, b_2)) = b_1b_2 \) is a continuous homomorphism of \( B_1 \times B_2 \) onto \( S \).

Suppose \( \psi((b_1, b_2)) = 0 \). Then \( b_1b_2 = 0 \). Suppose \( b_1 \neq 0 \neq b_2 \). Then, for each \( n \in N \), \( b_1^{1/n}b_2^{1/n} = 0 \). But \( b_1^{1/n} \to 1 \) and \( b_2^{1/n} \to 1 \). Thus \( 1 = 0 \).

This contradiction implies that either \( b_1 = 0 \) or \( b_2 = 0 \). Hence \( \psi((b_1, b_2)) = 0 \) if and only if \( (b_1, b_2) \in J \).

Suppose \( \psi((a, b)) = \psi((c, d)) \), \( (a, b), (c, d) \in (B_1 \times B_2)/J \). Then \( ab = cd \). Since \( B_1 \) and \( B_2 \) are U-semigroups, there exist \( p \in B_1 \) and \( q \in B_2 \) such that one of the following cases hold:

(i) \( a = cp \) and \( b = dq \);

(ii) \( a = cp \) and \( d = bq \);

(iii) \( c = ap \) and \( b = dq \);

(iv) \( c = ap \) and \( d = bq \).

We will assume that case (i) holds. The proof for the other cases is similar. Thus we have \( cp \cdot dq = cd \). Hence \( (pq)(cd) = cd \). Let \( x = pq \) and \( y = cd \). Then \( xy = y \). Hence, for each \( n \in N \), \( x^ny = y \). If \( x \neq 1 \), then \( x^n \to 0 \). Thus, if \( x \neq 1 \), then \( y = 0 \), and hence \( cd = 0 \).

By the preceding paragraph, either \( c = 0 \) or \( d = 0 \). But \( c \neq 0 \neq d \). Hence \( x = 1 \) and \( pq = 1 \). Then for each \( n \in N \), \( p^nsq^n = 1 \). If \( p \neq 1 \), \( p^n \to 0 \), and hence \( 0 = 1 \). Similarly, if \( q \neq 1 \), then \( 0 = 1 \). This contradiction implies that \( p = q = 1 \). Thus \( a = c, b = d \), and \( (a, b) = (c, d) \).

Hence \( \psi \) is one-to-one on \( (B_1B_2)/J \), thus completing the proof of the corollary.
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<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>M. J. C. Baker, <em>A spherical Helly-type theorem</em></td>
<td>1</td>
</tr>
<tr>
<td>Robert Morgan Brooks, *On locally m-convex <em>-algebras</em></td>
<td>5</td>
</tr>
<tr>
<td>Lindsay Nathan Childs and Frank Rimi DeMeyer, <em>On automorphisms of separable algebras</em></td>
<td>25</td>
</tr>
<tr>
<td>Charles L. Fefferman, <em>A Radon-Nikodym theorem for finitely additive set functions</em></td>
<td>35</td>
</tr>
<tr>
<td>Magnus Giertz, <em>On generalized elements with respect to linear operators</em></td>
<td>47</td>
</tr>
<tr>
<td>Mary Gray, <em>Abelian objects</em></td>
<td>69</td>
</tr>
<tr>
<td>Mary Gray, <em>Radical subcategories</em></td>
<td>79</td>
</tr>
<tr>
<td>John A. Hildebrant, <em>On uniquely divisible semigroups on the two-cell</em></td>
<td>91</td>
</tr>
<tr>
<td>Barry E. Johnson, <em>AW</em>-algebras are <em>QW</em>-algebras</td>
<td>97</td>
</tr>
<tr>
<td>Carl W. Kohls, <em>Decomposition spectra of rings of continuous functions</em></td>
<td>101</td>
</tr>
<tr>
<td>Calvin T. Long, <em>Addition theorems for sets of integers</em></td>
<td>107</td>
</tr>
<tr>
<td>Ralph David McWilliams, <em>On w</em>-sequential convergence and quasi-reflexivity*</td>
<td>113</td>
</tr>
<tr>
<td>Alfred Richard Mitchell and Roger W. Mitchell, <em>Disjoint basic subgroups</em></td>
<td>119</td>
</tr>
<tr>
<td>John Emanuel de Pillis, <em>Linear transformations which preserve hermitian and positive semidefinite operators</em></td>
<td>129</td>
</tr>
<tr>
<td>Qazi Ibadur Rahman and Q. G. Mohammad, <em>Remarks on Schwarz’s lemma</em></td>
<td>139</td>
</tr>
<tr>
<td>Neal Jules Rothman, <em>An L₁ algebra for certain locally compact topological semigroups</em></td>
<td>143</td>
</tr>
<tr>
<td>F. Dennis Sentilles, <em>Kernel representations of operators and their adjoints</em></td>
<td>153</td>
</tr>
<tr>
<td>D. R. Smart, <em>Fixed points in a class of sets</em></td>
<td>163</td>
</tr>
<tr>
<td>K. Srinivasacharyulu, <em>Topology of some Kähler manifolds</em></td>
<td>167</td>
</tr>
<tr>
<td>Francis C.Y. Tang, <em>On uniqueness of generalized direct decompositions</em></td>
<td>171</td>
</tr>
<tr>
<td>Albert Chapman Vosburg, <em>On the relationship between Hausdorff dimension and metric dimension</em></td>
<td>183</td>
</tr>
<tr>
<td>James Victor Whittaker, <em>Multiply transitive groups of transformations</em></td>
<td>189</td>
</tr>
</tbody>
</table>