

Pacific Journal of Mathematics

**DECOMPOSITION SPECTRA OF RINGS OF CONTINUOUS
FUNCTIONS**

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Let S be a subset of a completely regular Hausdorff space X . Sufficient conditions on S and X are obtained for the ring of continuous real-valued functions on S to be isomorphic to an inverse limit of quotient rings of the ring of continuous functions on X , or, alternatively, of the ring of bounded continuous functions on X . An application to the theory of rings of quotients of rings of continuous functions is given.

A decomposition spectrum of a set with some kind of structure is an inverse system of quotient structures of the same type. Decomposition spectra have been discussed recently by various authors: For topological spaces by Flachsmeyer [2], Pasynkov [4], and Vegrin [6]; and for ordered sets by Rinow [5]. Vegrin also considers briefly decomposition spectra of rings of continuous functions; however, the question he investigates is different from those considered here.

DEFINITION. A *decomposition spectrum* of a ring A is an inverse system of quotient rings of A .

The ring of all continuous real-valued functions on a completely regular Hausdorff space X will be denoted by $C(X)$, and the subring of bounded functions by $C^*(X)$.

The inverse limit of a decomposition spectrum of a ring is, of course, a ring. In the papers on decomposition spectra mentioned above, it often turns out that the inverse limit is an *extension* of the original structure (topological space or ordered set). Now if S is a subset of X , $C(S)$ is often an extension of $C(X)$; and $C(X)$ is always an extension of $C^*(X)$. This suggests the following questions: (1) For a subset S of X , when is $C(S)$ isomorphic to the inverse limit of some decomposition spectrum of $C(X)$? (2) When is $C(X)$ isomorphic to the inverse limit of some decomposition spectrum of $C^*(X)$?

The first question has the trivial answer: When S is C -embedded in X , that is when every function in $C(S)$ can be extended to a function in $C(X)$; for then, in fact, $C(S)$ is isomorphic to a quotient ring of $C(X)$, since the restriction mapping is a homomorphism onto $C(S)$. (This observation also leads naturally to the first question.) So some particular answers are: When S is compact, or when S is closed and X is normal, or when S is open-and-closed. The second question has the trivial answer: When X is compact; for in that case,

$C(X) = C^*(X)$. We show below that there are some nontrivial answers to both questions.

For any $f \in C(X)$ and any subset B of X , the restriction of f to B will be designated by $f|B$, and the image of B under f will be written $f[B]$. The constant function in $C(X)$ whose value is r will be denoted by r , and the greatest lower bound of f and g in the lattice $C(X)$ will be symbolized by $f \wedge g$. When we say that a collection $\{B_\gamma\}$ of subsets of a space Y containing a point p determines the topology of Y at p , we mean that if f is a real-valued function on Y and $f|B_\gamma$ is continuous for all B_γ , then f is continuous at p .

The following lemma is used in obtaining all of our results on decomposition spectra. A parallel statement to the one given explicitly is indicated by the symbols in square brackets.

LEMMA. *Let S be a subset of a completely regular Hausdorff space X . Suppose there exists a collection $\{T_\gamma\}$ of subsets of S with the following properties:*

- (1) $\{T_\gamma\}$ is closed under finite unions;
- (2) For each $p \in S$, the collection of all sets in $\{T_\gamma\}$ containing p determines the topology of S at p ;
- (3) For each $f \in C(S)$ and each T_γ , the function $f|T_\gamma$ can be extended to a function in $C(X)$ [$C^*(X)$].

Then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C(X)$ [$C^(X)$].*

Proof. One obtains the proof of the parallel statement by replacing " $C(X)$ " with " $C^*(X)$ " throughout the following proof.

From (1), $\{T_\gamma\}$ is directed by the relation \supset . For each γ , let I_γ be the ideal $\{h \in C(X) : h|T_\gamma = \{0\}\}$. Now each $C(X)/I_\gamma$ is isomorphic to $\{g|T_\gamma : g \in C(X)\}$, so we shall view each element of $C(X)/I_\gamma$ as an element of $\{g|T_\gamma : g \in C(X)\}$. Thus, if $T_\gamma \supset T_\delta$, then the natural homomorphism defined by $g|T_\gamma \rightarrow g|T_\delta$ for $g \in C(X)$ may be considered a homomorphism of $C(X)/I_\gamma$ onto $C(X)/I_\delta$. Also, the transitivity property is clearly satisfied by restriction mappings. Hence $\{C(X)/I_\gamma\}$ and the natural homomorphisms comprise a decomposition spectrum of $C(X)$.

Now let $f \in C(S)$ be given. We define an element $(f_\gamma) \in \lim (C(X)/I_\gamma)$ as follows: For each γ , f_γ is the image in $C(X)/I_\gamma$ of a function in $C(X)$ whose restriction to T_γ coincides with $f|T_\gamma$; the existence of such a function is ensured by (3). Then $(f_\gamma) \in \lim (C(X)/I_\gamma)$, because $T_\gamma \supset T_\delta$ implies $f_\gamma|T_\delta = (f|T_\gamma)|T_\delta = f|T_\delta = f_\delta$. The mapping $\sigma : f \rightarrow (f_\gamma)$ embeds $C(S)$ in $\lim (C(X)/I_\gamma)$, since $f \neq g$ implies $f(p) \neq g(p)$ for some $p \in S$, whence $f_\gamma \neq g_\gamma$ for any γ such that $p \in T_\gamma$. Furthermore, σ is

an isomorphism, because $(f + g)_\gamma = f_\gamma + g_\gamma$ and $(fg)_\gamma = f_\gamma g_\gamma$ for each γ .

To prove that σ is surjective, let $(b_\gamma) \in \lim(C(X)/I_\gamma)$ be given. Then b_γ is the restriction of a function in $C(\overleftarrow{X})$ to T_γ , and, since $\gamma < \delta$ implies that b_γ maps to b_δ under the natural homomorphism, b_γ is an extension of b_δ . By (2), $\{T_\gamma\}$ covers S , so (b_γ) may be associated with a function b on S . Since b is continuous on each T_γ , (2) implies that $b \in C(S)$; and $\sigma(b) = (b_\gamma)$.

THEOREM 1. *If X is a first countable space and S is any subset of X , then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C(X)$ [$C^*(X)$].*

Proof. Let $\{T_\gamma\}$ be the collection of all subsets of S consisting of a finite number of points of S together with sequences converging to those points. It is clear that (1) holds; (2) holds because X , and hence S , is first countable; and (3) holds because each T_γ is compact. Hence the Lemma is applicable.

COROLLARY 1. *If X is a first countable space, then $C(X)$ is isomorphic to the inverse limit of a decomposition spectrum of $C^*(X)$.*

THEOREM 2. *If X is a locally compact space, and S is any open subset of X , then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C^*(X)$.*

Proof. Let $\{T_\gamma\}$ be the collection of finite unions of some family of compact neighborhoods of the points of S . It is evident that (1) and (2) hold; and (3) holds because each T_γ is compact. Hence the Lemma is applicable.

COROLLARY 2. *If X is a locally compact space, then $C(X)$ is isomorphic to the inverse limit of a decomposition spectrum of $C^*(X)$.*

THEOREM 3. *If X is any completely regular Hausdorff space, and S is any open subset of X , then $C(S)$ is isomorphic to the inverse limit of a decomposition spectrum of $C(X)$.*

Proof. Let $p \in S$. By complete regularity, there exists an $h_p \in C(X)$ such that $h_p[X - S] = \{0\}$, $h_p(p) = 2$, and $0 \leq h_p \leq 2$. Hence the nonnegative function $g_p = h_p \wedge 1$ has the properties $g_p[X - S] = \{0\}$ and $g_p[U_p] = \{1\}$, where U_p is a neighborhood of p . Choose one such neighborhood for each point of S , and let $\{T_\gamma\}$ be the collection

of finite unions of these neighborhoods. Again, (1) and (2) are clear. To see that (3) holds, consider a particular $T_\gamma = U_{p_1} \cup \cdots \cup U_{p_n}$. The nonnegative function $g_{p_1} + \cdots + g_{p_n}$ is zero on $X - S$ and at least one on T_γ , whence $g_\gamma = (g_{p_1} + \cdots + g_{p_n}) \wedge 1$ has the properties $g_\gamma[X - S] = \{0\}$, $g_\gamma[T_\gamma] = \{1\}$, and $0 \leq g_\gamma \leq 1$. If $f \in C(S)$, we define $f_\gamma \in C(X)$ by $f_\gamma[X - S] = \{0\}$ and $f_\gamma(x) = \tan((g_\gamma(x))(\arctan(f(x))))$ for $x \in S$. Then $f_\gamma|_{T_\gamma} = (\tan \circ \arctan \circ f)|_{T_\gamma} = f|_{T_\gamma}$, as required. Thus (3) holds, and again the Lemma is applicable.

REMARK. If S is a cozero-set in X , say $S = \{x \in X: h(x) \neq 0\}$, where $h \in C(X)$, then the decomposition spectrum can be formed from an ω^* -sequence of quotient rings of $C(X)$. For each positive integer n , we set $T_n = \{x \in X: |h(x)| \geq 1/n\}$. There exists a function $g_n \in C(X)$ such that $g_n[X - S] = \{0\}$, $g_n[T_n] = \{1\}$, and $0 \leq g_n \leq 1$, since T_n is completely separated from $X - S$ [3; 1.15]. The proof that the collection $\{T_n\}$ satisfies (3) then concludes as in the proof of Theorem 3. Now (1) is evident, and (2) holds because each $p \in S$ is in the interior of some T_n ; so the Lemma is applicable to this situation too.

We now give an application of Theorem 3. First recall that the maximal ring of quotients of a commutative ring A with identity may be obtained as the direct limit of the A -homomorphisms of dense ideals of A into A [1; 1.9], and that the classical ring of quotients may be obtained similarly from the A -homomorphisms of dense principal ideals [1; 1.10]. Fine, Gillman, and Lambek have shown that (1) the maximal ring of quotients of $C(X)$ has a representation as the direct limit of rings $C(U)$, U ranging over the dense open sets in X , and that (2) the classical ring of quotients of $C(X)$ has a representation as the direct limit of rings $C(U)$, U ranging over the dense cozero-sets in X [1; 2.6]. Combining Theorem 3 with these facts yields the following result.

COROLLARY 3. *If X is any completely regular Hausdorff space, then both the maximal and classical rings of quotients of $C(X)$ have representations as a direct limit of inverse limits of quotient rings of $C(X)$.*

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Received January 3, 1966.

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Pacific Journal of Mathematics

Vol. 23, No. 1

March, 1967

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