ADDITION THEOREMS FOR SETS OF INTEGERS

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Let $C$ be a set of integers. Two subsets $A$ and $B$ of $C$ are said to be complementing subsets of $C$ in case every $c \in C$ is uniquely represented in the sum

$$C = A + B = \{x \mid x = a + b, a \in A, b \in B\}.$$ 

In this paper we characterize all pairs $A, B$ of complementing subsets of

$$N_n = \{0, 1, \ldots, n - 1\}$$

for every positive integer $n$ and show some interesting connections between these pairs and pairs of complementing subsets of the set $N$ of all nonnegative integers and the set $I$ of all integers. We also show that the number $C(n)$ of complementing subsets of $N_n$ is the same as the number of ordered nontrivial factorizations of $n$ and that

$$2C(n) = \sum_{d\mid n} C(d).$$

The structure of complementing pairs $A$ and $B$ has been studied by de Bruijn [1], [2], [3] for the cases $C = I$ and $C = N$ and by A. M. Vaidya [7] who reproduced a fundamental result of de Bruijn for the latter case. In case $C = N$ it is easy to see that $A \cap B = \{0\}$ and that $1 \in A \cup B$. Moreover, if we agree that $1 \in A$, it follows from the work of de Bruijn, that, except in the trivial case $A = N, B = \{0\}$, $A$ and $B$ are infinite complementing subsets of $N$ if and only if there exists an infinite sequence of integers $\{m_i\}_{i=1}^\infty$ with $m_i \geq 2$ for all $i$, such that $A$ and $B$ are the sets of all finite sums of the form

$$a = \sum x_i M_i,$$

$$b = \sum x_{i+1} M_{i+1},$$

respectively where $0 \leq x_i < m_{i+1}$ for $i \geq 0$ and where $M_0 = 1$ and $M_i = \prod_{j=1}^i m_j$ for $i \geq 1$. In the remaining case, when just one of $A$ and $B$ is infinite, the same result holds except that the sequence $\{m_i\}$ is of finite length $r$ and that $x_r \geq 0$. Similar results can also be obtained in the case of complementing $k$-tuples of subsets of $N$ for $k > 2$.

The case $C = I$ is much more difficult and, while sufficient conditions are easily given, necessary and sufficient conditions that a pair $A, B$ be complementing subsets of $I$ are not known. As an example of sufficient conditions, we note that if $A$ and $B$ are as in (1) above, then $A$ and $-B$ form a pair of complementing subsets of $I$. This is
an immediate consequence of the fact that every integer \( n \) can be represented uniquely in the form

\[
(2) \quad n = \sum_{i=0}^{r} (-1)^i x_i M_i
\]

with \( x_i \) and \( M_i \) as in (1). Incidentally, if \( B \) is finite, it is not difficult to see that there exists an integer \( r_0 \geq 0 \) such that \( A \) and \( -B \) form a pair of complementing subsets of the set

\[
R = \{ r \mid r \in I, r \geq r_0 \}.
\]

And if \( A \) is finite, there exists an integer \( s_0 > 0 \) such that \( A \) and \( -B \) are complementing subsets of the set

\[
S = \{ s \mid s \in I, s \leq s_0 \}.
\]

2. Complementing sets of order \( n \). We now investigate the structure of pairs \( A, B \) of complementing subsets of the set

\[
N_n = \{0, 1, \ldots, n - 1\}
\]

for integral values of \( n \geq 1 \). Such a pair of sets will be called complementing sets of order \( n \) and we will write \((A, B) \sim N_n\).

In case \( n = 1 \), we have only the trivial pair \( A = B = \{0\} \). For \( n > 1 \), it is easy to see that \( A \cap B = \{0\} \) and that \( 1 \in A \cup B \). We choose our notation so that \( 1 \in A \) and, if \( m \) is the least positive element in \( B \), then we also have that \( N_m \subset A \) and that none of \( m + 1, m + 2, \ldots, 2m - 1 \) appear in either \( A \) or \( B \). If \( B \) does not contain positive elements, we have only the trivial pair \( A = N_n, B = \{0\} \).

For the remainder of the paper, we restrict our attention to the case \( n > 1 \) and we use the notation \( mS \) to denote the set of all multiples of elements of a set \( S \) by an integer \( m \).

**Lemma 1.** Let \( A, B, C, \) and \( D \) be subsets of \( N_n \) such that, for a fixed integer \( m \geq 2 \),

\[
A = mC + N_m \quad \text{and} \quad B = mD.
\]

Then \((A, B) \sim N_{mp}\) if and only if \((C, D) \sim N_p\) where \( p \geq 1 \).

**Proof.** Suppose first that \((C, D) \sim N_p\). Then, for any \( s \in N_{mp} \), there exist integers \( q \in N_p \) and \( r \in N_m \) such that \( s = mq + r \). Since \((C, D) \sim N_p\), there exist \( c \in C \) and \( d \in D \) such that \( q = c + d \). But then

\[
s = m(c + d) + r = (mc + r) + md = a + b
\]

with \( a = mc + r \in A \) and \( b = md \in B \). Moreover, if this representation
is not unique, there exist $a' \in A, b' \in B, c' \in C, d' \in D,$ and $r' \in N_m$ such that
\[ s = a' + b' = (mc' + r') + md'. \]

But then $r = r'$ and
\[ c + d = q = c' + d' \]

and this violates the condition that $q$ be uniquely represented in the sum $C + D$.

Conversely, suppose that $(A, B) \sim N_m$. Then, for $s \in N_m$, there exist $a \in A, b \in B, c \in C, d \in D,$ and $r \in N_m$ such that
\[ sm = a + b = (mc + r) + md. \]

But this implies that $r = 0$ and that $s = c + d$. Also, if this representation of $s$ in $C + D$ is not unique, there exist $c' \in C$ and $d' \in D$ such that $s = c' + d'$. But then
\[ sm = cm + dm = c'm + d'm \]

and this violates the condition that $sm$ be uniquely represented in $A + B$.

The next lemma is an adaptation of a key result of de Bruijn [2, p. 16].

**Lemma 2.** If $(A, B) \sim N_n$, then there exist an integer $m \geq 2$ such that $m \mid n$ and a complementing pair $A', B'$ of order $n/m$, with $1 \in A'$ if $B \neq \{0\}$, such that
\[ \text{(3)} \quad A = mB' + N_m \quad \text{and} \quad B = mA'. \]

**Proof.** If $B = \{0\}$, then $A = N_n$ and the desired result follows with $A' = B' = \{0\}$ and $m = n$. If $B \neq \{0\}$, let $m$ be the least positive integer in $B$. Since $1 \in A$ and $A \cap B = \{0\}$, it follows that $m \geq 2$. Determine the integer $h$ such that
\[ hm \leq n < (h + 1)m. \]

Now the induction of de Bruijn's proof holds for all nonnegative integers less than $h$ and shows that all elements of $B$ less than $hm$ are multiples of $m$ and that, for each $k$ with $0 \leq k \leq h - 1$, the set
\[ \{km, km + 1, \ldots, km + m - 1\} \]

is either a subset of $A$ or is disjoint from $A$. This implies that $A'$ and $B'$ exist such that (1) holds and $1 \in A'$ provided we are able to show that $hm + r \in A \cup B$ for every integer $r \geq 0$. Contrariwise,
suppose that \( hm + r \in A \). Then \( hm + r + m \in A + B = N_n \), and this is impossible since \( hm + r + m \geq hm + m > n \). Similarly, if \( hm + r \in B \), then \( (m - 1) + hm + r \in A + B \) and we have the same contradiction. Thus (3) holds and it follows that \( m \) divides \( n \) and, by Lemma 1, that \((A', B') \sim N_{n/m}\).

The following theorem, which characterizes all complementing pairs of order \( n > 1 \), now follows by repeated application of Lemma 2.

**Theorem 1.** Sets \( A_i \) and \( B_i \) form a complementing pair of order \( n \geq 2 \) if and only if there exists a sequence \( \{m_i\}_{i=1}^r \) of integers not less than two such that

\[
 n = \sum_{i=1}^r m_i
\]

and such that \( A_i \) and \( B_i \) are the sets of all finite sums of the form

\[
a = \sum_{i=0}^{\lfloor (r-1)/2 \rfloor} x_{2i} M_{2i} \quad \text{and} \quad b = \sum_{i=0}^{\lfloor (r-2)/2 \rfloor} x_{2i+1} M_{2i+1}
\]

respectively with \( M_0 = 1 \), \( M_{i+1} = \prod_{j=1}^{i+1} = m_j \) and \( 0 \leq x_i < m_{i+1} \) for \( 0 \leq i < r \). If \( r = 1 \), we interpret the notation to mean that \( B_1 = \{0\} \).

It follows from Theorem 1 that there exists a one to one correspondence between the set \( \mathcal{C}_n \) of all pairs of complementing sets of order \( n > 1 \) and the set of all ordered finite sequences \( \{m_i\} \) with \( m_i \geq 2 \) such that \( \prod m_i = n \). Thus, if \( C(n) \) denotes the number of elements of \( \mathcal{C}_n \), then \( C(n) \) is equal to the number \( F(n) \) of ordered nontrivial factorizations of \( n \). Curiously, as shown by P. A. MacMahon [4; p. 108], \( F(n) \) is in turn equal to the number of perfect partitions of \( n - 1 \). This last result is also listed by Riordan [6; pp. 123-4]. In a second paper, MacMahon [5; pp. 843-4] shows that

\[
 C(n) = \sum_{j=1}^q \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \sum_{\alpha_h} \binom{\alpha_h + j - i - 1}{\alpha_h}
\]

where \( q = \sum_{h=1}^\infty \alpha_h \) and \( n = \prod_{h=1} p_h^{\alpha_h} \) is the canonical representation of \( n \). However, if one actually wants the values of \( C(n) \), they are much more easily computed using the result of the following theorem:

**Theorem 2.** If \( n > 1 \) is an integer, then

\[
 C(n) = \frac{1}{2} \sum_{d|n} C(d) = 2 \sum_{d|n} \mu(d) C(n/d)
\]

where \( \mu \) denotes the Möbius function.
Proof. It follows from Lemma 2 that to each of the \( C(n) \) distinct complementing pairs \( A, B \) of order \( n \) there corresponds a unique complementing pair \( A', B' \) of order \( d \) where \( d \mid n \) and \( 1 \leq d < n \). Hence,

\[
C(n) \leq \sum_{d \mid n, d < n} C(d) .
\]

Moreover, from each of the \( C(d) \) distinct complementing pairs \( C, D \) of order \( d \), with \( 1 \leq d < n \) and \( 1 \in D \) if \( d \neq 1 \), can be formed precisely one pair \( A, B \) of complementing sets of order \( dq = n \) by the method of Lemma 1. Since the new pairs formed in this way are clearly distinct, it follows that

\[
C(n) \geq \sum_{d \mid n, d < n} C(d) .
\]

Thus, equality holds and this implies that

\[
C(n) = \frac{1}{2} \sum_{d \mid n} C(d)
\]

as claimed. The other equality is an immediate consequence of the Möbius inversion formula.

Except for Theorem 2, the preceding theorems reveal a striking parallel between the structure of complementing subsets of \( N \) and the structure of complementing pairs of order \( n \). The next theorem exhibits an additional interesting connecting between these two classes of pairs. Also, it is clear that a similar theorem holds giving sufficient conditions that \( A \) and \( B \) form a pair of complementing subsets of \( I \).

**Theorem 3.** Let \( \{m_i\}_{i \geq 1} \) and \( \{M_i\}_{i \geq 0} \) be as defined in (1) above and let \( (C_i, D_i) \sim N_{m_i+1} \) for \( i \geq 0 \). If \( A \) and \( B \) are the sets of all finite sums of the form

\[
a = \sum c_i M_i \quad \text{and} \quad b = \sum d_i M_i
\]

respectively with \( c_i \in C_i \) and \( d_i \in D_i \) for \( i \geq 0 \), then \( (A, B) \sim N \).

**Proof.** Let \( n \) be any nonnegative integer. Then \( n \) can be represented uniquely in the form

\[
n = \sum_{i=0}^{r} e_i M_i
\]

with \( e_i \in N_{m_i+1} \) for all \( i \). Since \( (C_i, D_i) \sim N_{m_i+1} \), there exist \( c_i \in C_i \) and \( d_i \in D_i \) such that \( e_i = c_i + d_i \) uniquely. Therefore,

\[
n = \sum_{i=0}^{r} (c_i + d_i) M_i
\]

\[
= \sum_{i=0}^{r} c_i M_i + \sum_{i=0}^{r} d_i M_i
\]

\[
= a + b
\]
with \( a \in A \) and \( b \in B \). If this representation of \( n \) in \( A + B \) is not unique, there exist \( a' \in A \) and \( b' \in B \) such that

\[
    n = a' + b'
\]

where

\[
    a' = \sum_{i=0}^{s} c_i M_i \quad \text{and} \quad b' = \sum_{i=0}^{s} d_i M_i
\]

with \( c_i \in C_i \) and \( d_i \in D_i \) for each \( i \). But then

\[
    n = \sum_{i=0}^{s} (c_i' + d_i')M_i
\]

and \( c_i' + d_i' \in N_{m_{i+1}} \) since \((C_i, D_i) \sim N_{m_{i+1}}\) for all \( i \). Since representations of \( n \) in this form are unique, it follows that \( r = s \) and that

\[
    c_i + d_i = c_i' + d_i'
\]

for each \( i \). And this violates the condition that \((C_i, D_i) \sim N_{m_{i+1}}\). Thus, the representation is unique and \((A, B) \sim N\) as claimed.

Note that if \( r \) is fixed and \( 0 \leq i < r \) in the sums defining \( A \) and \( B \) in the preceding theorem, then we conclude in the same way that \((A, B) \sim N_r\).

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