ADDITION THEOREMS FOR SETS OF INTEGERS

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Let $C$ be a set of integers. Two subsets $A$ and $B$ of $C$ are said to be complementing subsets of $C$ in case every $c \in C$ is uniquely represented in the sum

$$C = A + B = \{x | x = a + b, a \in A, b \in B\}.$$ 

In this paper we characterize all pairs $A, B$ of complementing subsets of

$$N_n = \{0, 1, \cdots, n - 1\}$$

for every positive integer $n$ and show some interesting connections between these pairs and pairs of complementing subsets of the set $N$ of all nonnegative integers and the set $I$ of all integers. We also show that the number $C(n)$ of complementing subsets of $N_n$ is the same as the number of ordered nontrivial factorizations of $n$ and that

$$2C(n) = \sum_{d | n} C(d).$$

The structure of complementing pairs $A$ and $B$ has been studied by de Bruijn [1], [2], [3] for the cases $C = I$ and $C = N$ and by A. M. Vaidya [7] who reproduced a fundamental result of de Bruijn for the latter case. In case $C = N$ it is easy to see that $A \cap B = \{0\}$ and that $1 \in A \cup B$. Moreover, if we agree that $1 \in A$, it follows from the work of de Bruijn, that, except in the trivial case $A = N, B = \{0\}$, $A$ and $B$ are infinite complementing subsets of $N$ if and only if there exists an infinite sequence of integers $\{m_i\}_{i=1}^\infty$ with $m_i \geq 2$ for all $i$, such that $A$ and $B$ are the sets of all finite sums of the form

\begin{align*}
    a &= \sum x_{2i}M_{2i}, \\
    b &= \sum x_{2i+1}M_{2i+1}
\end{align*}

(1)

respectively where $0 \leq x_i < m_{i+1}$ for $i \geq 0$ and where $M_0 = 1$ and $M_i = \prod_{j=1}^i m_j$ for $i \geq 1$. In the remaining case, when just one of $A$ and $B$ is infinite, the same result holds except that the sequence $\{m_i\}$ is of finite length $r$ and that $x_r \geq 0$. Similar results can also be obtained in the case of complementing $k$-tuples of subsets of $N$ for $k > 2$.

The case $C = I$ is much more difficult and, while sufficient conditions are easily given, necessary and sufficient conditions that a pair $A, B$ be complementing subsets of $I$ are not known. As an example of sufficient conditions, we note that if $A$ and $B$ are as in (1) above, then $A$ and $-B$ form a pair of complementing subsets of $I$. This is
an immediate consequence of the fact that every integer \( n \) can be represented uniquely in the form

\[
n = \sum_{i=0}^{r} (-1)^i x_i M_i
\]

with \( x_i \) and \( M_i \) as in (1). Incidentally, if \( B \) is finite, it is not difficult to see that there exists an integer \( r_0 \leq 0 \) such that \( A \) and \(-B\) form a pair of complementing subsets of the set

\[
R = \{ r \mid r \in I, r \geq r_0 \}.
\]

And if \( A \) is finite, there exists an integer \( s_0 > 0 \) such that \( A \) and \(-B\) are complementing subsets of the set

\[
S = \{ s \mid s \in I, s \leq s_0 \}.
\]

2. Complementing sets of order \( n \). We now investigate the structure of pairs \( A, B \) of complementing subsets of the set

\[
N_n = \{0, 1, \ldots, n-1\}
\]

for integral values of \( n \geq 1 \). Such a pair of sets will be called complementing sets of order \( n \) and we will write \((A, B) \sim N_n\).

In case \( n = 1 \), we have only the trivial pair \( A = B = \{0\} \). For \( n > 1 \), it is easy to see that \( A \cap B = \{0\} \) and that \( 1 \in A \cup B \). We choose our notation so that \( 1 \in A \) and, if \( m \) is the least positive element in \( B \), then we also have that \( N_m \subset A \) and that none of \( m + 1, m + 2, \ldots, 2m - 1 \) appear in either \( A \) or \( B \). If \( B \) does not contain positive elements, we have only the trivial pair \( A = N_n, B = \{0\} \).

For the remainder of the paper, we restrict our attention to the case \( n > 1 \) and we use the notation \( mS \) to denote the set of all multiples of elements of a set \( S \) by an integer \( m \).

**Lemma 1.** Let \( A, B, C, \) and \( D \) be subsets of \( N_n \) such that, for a fixed integer \( m \geq 2 \),

\[
A = mC + N_m \quad \text{and} \quad B = mD.
\]

Then \((A, B) \sim N_{mp}\) if and only if \((C, D) \sim N_p\) where \( p \geq 1 \).

**Proof.** Suppose first that \((C, D) \sim N_p\). Then, for any \( s \in N_{mp} \), there exist integers \( q \in N_p \) and \( r \in N_m \) such that \( s = mq + r \). Since \((C, D) \sim N_p\), there exist \( c \in C \) and \( d \in D \) such that \( q = c + d \). But then

\[
s = m(c + d) + r = (mc + r) + md = a + b
\]

with \( a = mc + r \in A \) and \( b = md \in B \). Moreover, if this representation
is not unique, there exist \( a' \in A, b' \in B, c' \in C, d' \in D, \) and \( r' \in N_m \) such that
\[
s = a' + b' = (mc' + r') + md'.
\]
But then \( r = r' \) and
\[
c + d = q = c' + d'
\]
and this violates the condition that \( q \) be uniquely represented in the sum \( C + D \).

Conversely, suppose that \( (A, B) \sim N_m \). Then, for \( s \in N_p \), there exist \( a \in A, b \in B, c \in C, d \in D, \) and \( r \in N_m \) such that
\[
sm = a + b = (mc + r) + md.
\]
But this implies that \( r = 0 \) and that \( s = c + d \). Also, if this representation of \( s \) in \( C + D \) is not unique, there exist \( c' \in C \) and \( d' \in D \) such that \( s = c' + d' \). But then
\[
sm = cm + dm = c'm + d'm
\]
and this violates the condition that \( sm \) be uniquely represented in \( A + B \).

The next lemma is an adaptation of a key result of de Bruijn [2, p. 16].

**Lemma 2.** If \( (A, B) \sim N_n \), then there exist an integer \( m \geq 2 \) such that \( m \mid n \) and a complementing pair \( A', B' \) of order \( n/m \), with \( 1 \in A' \) if \( B \neq \{0\} \), such that
\[
(3) \quad A = mB' + N_m \quad \text{and} \quad B = mA'.
\]

**Proof.** If \( B = \{0\} \), then \( A = N_n \) and the desired result follows with \( A' = B' = \{0\} \) and \( m = n \). If \( B \neq \{0\} \), let \( m \) be the least positive integer in \( B \). Since \( 1 \in A \) and \( A \cap B = \{0\} \), it follows that \( m \geq 2 \). Determine the integer \( h \) such that
\[
hm \leq n < (h + 1)m.
\]
Now the induction of de Bruijn's proof holds for all nonnegative integers less than \( h \) and shows that all elements of \( B \) less than \( hm \) are multiples of \( m \) and that, for each \( k \) with \( 0 \leq k \leq h - 1 \), the set
\[
\{km, km + 1, \ldots, km + m - 1\}
\]
is either a subset of \( A \) or is disjoint from \( A \). This implies that \( A' \) and \( B' \) exist such that \( (1) \) holds and \( 1 \in A' \) provided we are able to show that \( hm + r \in A \cup B \) for every integer \( r \geq 0 \). Contrariwise,
suppose that \(hm + r \in A\). Then \(hm + r + m \in A + B = N_n\), and this is impossible since \(hm + r + m \geq hm + m > n\). Similarly, if \(hm + r \in B\), then \((m - 1) + hm + r \in A + B\) and we have the same contradiction. Thus (3) holds and it follows that \(m\) divides \(n\) and, by Lemma 1, that \((A', B') \sim N_{n/m}\).

The following theorem, which characterizes all complementing pairs of order \(n > 1\), now follows by repeated application of Lemma 2.

**Theorem 1.** Sets \(A_i\) and \(B_i\) form a complementing pair of order \(n \geq 2\) if and only if there exists a sequence \(\{m_i\}_{i=1}^{r}\) of integers not less than two such that

\[
n = \sum_{i=1}^{r} m_i
\]

and such that \(A_i\) and \(B_i\) are the sets of all finite sums of the form

\[
a = \sum_{i=0}^{\lceil (r-1)/2 \rceil} x_{2i} M_{2i}
\text{ and } b = \sum_{i=0}^{\lceil (r-2)/2 \rceil} x_{2i+1} M_{2i+1}
\]

respectively with \(M_0 = 1\), \(M_{i+1} = \Pi_{j=1}^{i+1} m_j\) and \(0 \leq x_i < m_{i+1}\) for \(0 \leq i < r\). If \(r = 1\), we interpret the notation to mean that \(B_1 = \{0\}\).

It follows from Theorem 1 that there exists a one to one correspondence between the set \(\mathcal{C}_n\) of all pairs of complementing sets of order \(n > 1\) and the set of all ordered finite sequences \(\{m_i\}\) with \(m_i \geq 2\) such that \(\Pi m_i = n\). Thus, if \(C(n)\) denotes the number of elements of \(\mathcal{C}_n\), then \(C(n)\) is equal to the number \(F(n)\) of ordered nontrivial factorizations of \(n\). Curiously, as shown by P. A. MacMahon [4; p. 108], \(F(n)\) is in turn equal to the number of perfect partitions of \(n-1\). This last result is also listed by Riordan [6; pp. 123-4]. In a second paper, MacMahon [5; pp. 843-4] shows that

\[
C(n) = \sum_{j=1}^{q} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \sum_{h=1}^{r} \binom{\alpha_h + j - i - 1}{\alpha_h - 1}
\]

where \(q = \sum_{h=1}^{r} \alpha_h\) and \(n = \Pi_{h=1}^{r} p_h^{\alpha_h}\) is the canonical representation of \(n\). However, if one actually wants the values of \(C(n)\), they are much more easily computed using the result of the following theorem:

**Theorem 2.** If \(n > 1\) is an integer, then

\[
C(n) = \frac{1}{2} \sum_{d|n} C(d) = 2 \sum_{d|n} \mu(d)C(n/d)
\]

where \(\mu\) denotes the Möbius function.
Proof. It follows from Lemma 2 that to each of the \( C(n) \) distinct complementing pairs \( A, B \) of order \( n \) there corresponds a unique complementing pair \( A', B' \) of order \( d \) where \( d \mid n \) and \( 1 \leq d < n \). Hence,

\[
C(n) \leq \sum_{d \mid n, d < n} C(d).
\]

Moreover, from each of the \( C(d) \) distinct complementing pairs \( C, D \) of order \( d \), with \( 1 \leq d < n \) and \( 1 \in D \) if \( d \neq 1 \), can be formed precisely one pair \( A, B \) of complementing sets of order \( dq = n \) by the method of Lemma 1. Since the new pairs formed in this way are clearly distinct, it follows that

\[
C(n) \geq \sum_{d \mid n, d < n} C(d).
\]

Thus, equality holds and this implies that

\[
C(n) = \frac{1}{2} \sum_{d \mid n} C(d)
\]
as claimed. The other equality is an immediate consequence of the Möbius inversion formula.

Except for Theorem 2, the preceding theorems reveal a striking parallel between the structure of complementing subsets of \( N \) and the structure of complementing pairs of order \( n \). The next theorem exhibits an additional interesting connecting between these two classes of pairs. Also, it is clear that a similar theorem holds giving sufficient conditions that \( A \) and \( B \) form a pair of complementing subsets of \( I \).

**Theorem 3.** Let \( \{m_i\}_{i \geq 1} \) and \( \{M_i\}_{i \geq 0} \) be as defined in (1) above and let \( (C_i, D_i) \sim N_{m_i+1} \) for \( i \geq 0 \). If \( A \) and \( B \) are the sets of all finite sums of the form

\[
a = \sum c_i M_i \quad \text{and} \quad b = \sum d_i M_i
\]

respectively with \( c_i \in C_i \) and \( d_i \in D_i \) for \( i \geq 0 \), then \( (A, B) \sim N \).

**Proof.** Let \( n \) be any nonnegative integer. Then \( n \) can be represented uniquely in the form

\[
n = \sum_{i=0}^{r} e_i M_i
\]

with \( e_i \in N_{m_i+1} \) for all \( i \). Since \( (C_i, D_i) \sim N_{m_i+1} \), there exist \( c_i \in C_i \) and \( d_i \in D_i \) such that \( e_i = c_i + d_i \) uniquely. Therefore,

\[
n = \sum_{i=0}^{r} (c_i + d_i) M_i
\]

\[
= \sum_{i=0}^{r} c_i M_i + \sum_{i=0}^{r} d_i M_i
\]

\[
= a + b
\]
with \( a \in A \) and \( b \in B \). If this representation of \( n \) in \( A + B \) is not unique, there exist \( a' \in A \) and \( b' \in B \) such that

\[
n = a' + b'
\]

where

\[
a' = \sum_{i=0}^{s} c_i' M_i \quad \text{and} \quad b' = \sum_{i=0}^{s} d_i' M_i
\]

with \( c_i' \in C_i \) and \( d_i' \in D_i \) for each \( i \). But then

\[
n = \sum_{i=0}^{s} (c_i' + d_i') M_i
\]

and \( c_i' + d_i' \in N_{m_i+1} \) since \((C_i, D_i) \sim N_{m_i+1} \) for all \( i \). Since representations of \( n \) in this form are unique, it follows that \( r = s \) and that

\[
c_i + d_i = c_i' + d_i'
\]

for each \( i \). And this violates the condition that \((C_i, D_i) \sim N_{m_i+1} \).

Thus, the representation is unique and \((A, B) \sim N \) as claimed.

Note that if \( r \) is fixed and \( 0 \leq i < r \) in the sums defining \( A \) and \( B \) in the preceding theorem, then we conclude in the same way that \((A, B) \sim N_{n^r} \).

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WASHINGTON STATE UNIVERSITY
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Robert Morgan Brooks, *On locally m-convex *-algebras* ........................................ 5
Lindsay Nathan Childs and Frank Rimi DeMeyer, *On automorphisms of separable algebras* ................................................................. 25
Charles L. Fefferman, *A Radon-Nikodym theorem for finitely additive set functions* ................................................................. 35
Magnus Giertz, *On generalized elements with respect to linear operators* .................. 47
Mary Gray, *Abelian objects* ....................................................................................... 69
Mary Gray, *Radical subcategories* ............................................................................... 79
John A. Hildebrant, *On uniquely divisible semigroups on the two-cell* .................. 91
Barry E. Johnson, *AW*-algebras are *QW*-algebras .................................................. 97
Carl W. Kohls, *Decomposition spectra of rings of continuous functions* ............... 101
Calvin T. Long, *Addition theorems for sets of integers* ........................................... 107
Ralph David McWilliams, *On w*-sequential convergence and quasi-reflexivity ........ 113
Alfred Richard Mitchell and Roger W. Mitchell, *Disjoint basic subgroups* .......... 119
John Emanuel de Pillis, *Linear transformations which preserve hermitian and positive semidefinite operators* ................................................... 129
Qazi Ibadur Rahman and Q. G. Mohammad, *Remarks on Schwarz’s lemma* .......... 139
Neal Jules Rothman, *An L 1 algebra for certain locally compact topological semigroups* ......................................................................................... 143
F. Dennis Sentilles, *Kernel representations of operators and their adjoints* ............... 153
D. R. Smart, *Fixed points in a class of sets* ............................................................... 163
K. Srinivasacharyulu, *Topology of some Kähler manifolds* ....................................... 167
Francis C.Y. Tang, *On uniqueness of generalized direct decompositions* ............. 171
Albert Chapman Vosburg, *On the relationship between Hausdorff dimension and metric dimension* ......................................................... 183
James Victor Whittaker, *Multiply transitive groups of transformations* ................. 189